QUANTILE ESTIMATION USING THE MAXIMUM TRANSFORMATION, STOCHASTIC APPROXIMATION AND THE JACKKNIFE

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THESIS

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ABSTRACT

The rate of convergence of the expected value of quantile estimates using a stochastic approximation with the maximum transformation is evaluated. The analysis is performed using linear regression techniques on computer simulation results for quantile estimates of the unit exponential distribution. Included is a discussion on the use of the jackknife technique to reduce the bias of the stochastic approximation quantile estimates. Simulation results for the 2-fold jackknife for the $m^{-.5}$ term are tabulated. The main conclusion of the analysis is that the lowest order term in the expression for the expected value of the estimate as a function of sample size decreases as $m^{-.25}$.

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I. INTRODUCTION

The problem of estimating quantiles may be found in numerous statistical applications. (The α quantile S_α for the distribution of the random variable S, F(s), is defined as $S_\alpha = F^{-1}(\alpha)$, $0 \leq \alpha \leq 1$, and is assumed to be unique). The increased use of computer system simulations provides some practical examples. In certain cases the quantile estimates from the generated data may be the direct measures used to evaluate the system. In other applications one might find it practical to characterize the distribution function of variables at different points in a system using several statistics such as quantiles. The practicability of this latter example may be dictated by the size of the simulation and the number of replications performed. In any case, the need for efficient quantile estimates exists.

Two possible solutions to the problem of quantile estimation are the order statistic estimate ($\bar{S}_\alpha(m)$) and the stochastic approximation estimate ($\hat{S}_\alpha(m)$) of Robbins-Monro (1951) where m is the sample size. These methods each have their benefits and limitations which warrant comparison. In this regard, the increased use of computer simulations dictates that the comparative efficiency of these methods not be based solely on statistical efficiency but computational efficiency as well. The term computational efficiency refers to the time and cost of programming,

debugging, computing, and data storage requirements in the application of a given estimation scheme.

Goodman, Lewis and Robbins (1973) used these considerations in a comparative analysis of the order statistic estimate and the stochastic approximation estimate with the maximum (minimum) transformation in estimating extreme quantiles (e.g., quantiles corresponding to probability levels $\alpha = 0.999$). The maximum transformation is a device which they utilized to alleviate the problem of the apparently very slow convergence of the stochastic approximation when applied to extreme quantiles. They also indicated the possible existence of similar problems in highly skewed distributions even for moderate quantiles (e.g., $\alpha = 0.90$).

The considerations for selecting the maximum transformation above other possible techniques are discussed by Goodman, Lewis, and Robbins (1973). The benefits are that it does not require special information about the random variable from which the sample is taken to estimate the quantile. This kind of information would be required in deriving bounds to the quantile estimate to alleviate the problem of very slow convergence. The maximum transformation is also computationally efficient in that it requires a single memory cell and a limited number of binary comparisons. These requirements are much simpler and less time consuming than the scheme proposed by Kesten (1958) to alleviate the problem of long runs. Finally, if the maximum transformation to the median ($\alpha' = 0.5$) is used,

Cochran and Davis (1965) have shown (using simulations) that the stochastic approximation works well for this value of α'. The maximum transformation is used to transform higher quantiles $\alpha > 0.5$ to $\alpha' = 0.5$ and the minimum transformation used where $\alpha < 0.5$.

The conclusions of Goodman, Lewis, and Robbins (1973) were that the stochastic approximation with maximum transformation provides a practical alternative to the problem of extreme quantile estimation. It is computationally efficient; computing time is linearly dependent on sample size (m) and storage requirements are small and fixed. Statistically, the estimate was relatively free of bias although there existed some increase in the variance of the estimate. Also included in their report were the results of the application of the jackknife for a presumed m^{-1} term in the bias to the stochastic approximation scheme. Their simulation results indicated that convergence was slower than m^{-1} and they conjectured the possible existence of an $m^{-.5}$ bias term. This implied the possibility of further improving the stochastic approximation with maximum transformation using the appropriate jackknifing procedure. It was this conjecture that prompted the analysis for this paper.

II. PROBLEM AND APPROACH

The problem approached was to evaluate the rate of convergence of the stochastic approximation with a maximum transformation with the intent to further improve the stochastic approximation with the jackknife method of bias reduction proposed by Quenouille (1956). A secondary problem was to evaluate Turkey's assertion that the sample variance of the "pseudo-values" used in forming the jackknifed estimate could be used to estimate the variance of the estimate, i.e., $\mathrm{Var}(\hat{S}_\alpha(m))$.

The approach taken to solve the problem consisted of two basic steps. Initially, a preliminary analysis was made on the data obtained by Goodman, Lewis, and Robbins (1973) to determine if the bias was of the order $0(m^{-.5})$ rather than $0(m^{-1})$. The linear least square computer program of Daniel and Wood (1971) was used to accomplish this analysis. Once the order of the bias term was determined, a computer simulation was used to evaluate the effect of the jackknife on the rate of convergence and the variance of the stochastic approximation estimate with maximum transformation and to determine if the sample variance of the pseudo-values could be used to estimate the variance of the quantile estimate. The distributions and quantiles selected for analysis were the unit exponential 0.5, 0.9, 0.95, 0.975, and 0.999 quantiles and the uniform (0,1) 0.5 quantile. Initially, the simulation was programmed to

achieve an m-fold jackknifed estimate but was subsequently modified to the 2-fold jackknife for reasons discussed later in the paper.

III. LINEAR REGRESSION

The preliminary analysis in evaluating the rate of convergence of the stochastic approximation with maximum transformation was accomplished using a regression analysis on the original data obtained by Goodman, Lewis, and Robbins (1973). This data consisted of 12 points covering reduced sample size values from $m' = 8$ to 60, where $m' = m/v$ and v is a constant computed for the maximum transformation, with corresponding values of the mean and variance of the stochastic approximation estimates with maximum transformation both with and without jackknife. The data used for the regression analysis was for the 0.999 quantile estimates of the unit exponential distribution.

Two equations were fitted to the data to substantiate the existence of a term of lower order than m^{-1}. In each case a weighted linear regression was used to account for the variability of the variance over the range of sample values. The weighting factor used was the inverse of the sample variance.

The initial regression was applied based on a logarithmic transformation. The regression equation used was

$$\ln(\text{bias}) = \ln(a) - \ln(m') \ , \qquad (3.1)$$

where the values of the bias were computed as

$$\hat{b} = E(\hat{S}_\alpha(m')) - S_\alpha \qquad (3.2)$$

and

$$\hat{b} = \tilde{\tilde{E}}(S_\alpha(m')) - S_\alpha \tag{3.3}$$

for the data without and with jackknife respectively. If the bias is such that $b = am^{-\gamma}$ and higher order terms can be neglected, then the coefficients obtained from the regression will provide a direct estimate of γ. An obvious problem exists if the bias contains secondary terms that are significant. In that case the estimate $\hat{\gamma}$ will be incorrect since the effect of secondary terms are not distinguishable under the logarithmic transformation. Consequently, a second equation was also applied using multiple independent variables to check the significance of any secondary terms.

The second regression equation used was

$$\text{bias} = am^{-\gamma} + bm^{-2\gamma} + cm^{-3\gamma} , \tag{3.4}$$

where γ is considered a known constant to preclude the requirement for a non-linear regression. The inherent problem with applying equation 3.4 with a linear regression is that an estimate or guess of γ is required. To achieve this estimate the results of the initial regression in conjunction with the conjecture that $\gamma = 0.5$ were evaluated to derive an estimate for implementing the regression using Equation 3.4.

IV. JACKKNIFE

Based on the assumption that the bias of the quantile estimate $\hat{S}_\alpha(m)$ is of the form

$$E(\hat{S}_\alpha) - S_\alpha = am^{-\gamma} + bm^{-2\gamma} + 0(m^{-3\gamma}), \tag{4.1}$$

the jackknife technique can be applied to obtain an estimate $\tilde{\tilde{S}}_\alpha(m)$ with a bias free of the $m^{-\gamma}$ term:

$$E(\tilde{\tilde{S}}_\alpha) - S_\alpha = b'm^{-2\gamma} + 0(m^{-3\gamma}) . \tag{4.2}$$

The initial approach to applying the technique was with the m-fold jackknife. Let $\hat{S}_\alpha(m)$ be the estimate based on all m sample values and $\hat{S}_\alpha(m-1)_i$, $i=1,\ldots,m$, be the estimate based on m-1 sample values obtained by omitting the ith value. Then consider the the pseudo-values $\tilde{S}_\alpha(m)_i$, $i=1,\ldots,m$, computed as

$$\tilde{S}_\alpha(m)_i = A \cdot \hat{S}_\alpha(m) + B \cdot \hat{S}_\alpha(m-1)_i \qquad (i=1,\ldots,m). \tag{4.3}$$

The jackknifed estimate $\tilde{\tilde{S}}_\alpha$ is then computed as the average of the m pseudo values:

$$\tilde{\tilde{S}}_\alpha = E(\tilde{S}_\alpha(m)) = \sum_{i=1}^{m} \tilde{S}_\alpha(m)_i/m. \tag{4.4}$$

From equations 4.1, 4.3, and 4.4, if A and B are selected such that

$$A = \frac{m^\gamma}{m^\gamma - (m-1)^\gamma} \qquad \text{and} \qquad B = -\frac{(m-1)^\gamma}{m^\gamma - (m-1)^\gamma}, \tag{4.5}$$

then the jackknifed estimate will have a bias which is $O(m^{-2\gamma})$, or

$$E(\tilde{\tilde{S}}_\alpha) - S_\alpha = - \frac{b}{m^\gamma (m-1)^\gamma} + O(m^{-3\gamma}). \qquad (4.6)$$

In the present case where γ is assumed to be 0.5 the values of A and B used in the simulation were

$$A = \frac{\sqrt{m}}{\sqrt{m} - \sqrt{(m-1)}} \quad \text{and} \quad B = - \frac{\sqrt{(m-1)}}{\sqrt{m} - \sqrt{(m-1)}} . \qquad (4.7)$$

The 2-fold jackknife is computed with the modification that the sample is partitioned into two disjoint sections with $\hat{S}_\alpha(m/2)_i$, i=1,2, computed from the ith section of half the sample values. Equations 4.3 and 4.4 then reduce to

$$\tilde{S}_\alpha(m)_1 = A \cdot \hat{S}_\alpha(m) + B \cdot \hat{S}_\alpha(m/2)_1 , \qquad (4.8)$$

$$\tilde{S}_\alpha(m)_2 = A \cdot \hat{S}_\alpha(m) + B \cdot \hat{S}_\alpha(m/2)_2 , \qquad (4.9)$$

and

$$\tilde{\tilde{S}}_\alpha = \frac{(\tilde{S}_\alpha(m)_1 + \tilde{S}_\alpha(m)_2)}{2} . \qquad (4.10)$$

There is a problem here in that it is possible to segment the sample into two disjoint sections in many ways; in what follows the sections were the first and second halves of the data.

The values of A and B required to achieve the reduction in bias then become

$$A = \frac{2^\gamma}{(2^\gamma - 1)} \quad \text{and} \quad B = - \frac{1}{(2^\gamma - 1)} . \qquad (4.11)$$

The resulting jackknifed estimate then is biased as $0(m^{-2\gamma})$, or specifically

$$E(\tilde{\tilde{S}}_\alpha) - S_\alpha = \frac{(2^\gamma - 2^{2\gamma})}{(2^\gamma - 1)} (b/m^{2\gamma}) + 0(m^{-3\gamma}) . \tag{4.12}$$

Based on the assumption $\gamma = 0.5$ the values for A and B are

$$A = \frac{\sqrt{2}}{(\sqrt{2} - 1)} \quad \text{and} \quad B = - \frac{1}{(\sqrt{2} - 1)} , \tag{4.13}$$

with the result that the jackknifed estimate has the form

$$E(\tilde{\tilde{S}}_\alpha) = S_\alpha + \frac{(\sqrt{2} - 2)}{(\sqrt{2} - 1)} (b/m) + 0(m^{-1.5}) . \tag{4.14}$$

These calculations indicate that if the jackknife is performed for the correct term, the resulting estimate should converge more rapidly but with a negative bias.

A logical question that arises is what are the consequences of applying the jackknife for an erroneous term, say $m^{-\beta}$. Considering the true bias to be of the form in equation 4.1 and applying the jackknife with $A = 2^\beta/(2^\beta - 1)$ and $B = -1/(2^\beta - 1)$ the resulting estimate will have the form

$$E(\tilde{\tilde{S}}_\alpha) = S_\alpha + \frac{(2^\beta - 2^\gamma)}{(2^\beta - 1)} (b/m^\gamma) + 0\ (m^{-2\gamma}) . \tag{4.15}$$

This result indicates that if the true bias is of the form $m^{-\gamma}$ such that $\gamma > \beta$, where β is the power of the term being

jackknifed, the resulting estimator should remain biased with leading term still decreasing as $m^{-\gamma}$ but with a coefficient $k = (2^{\beta} - 2^{\gamma})/(2^{\beta} - 1) < 0$. Similarly, if γ and β are such that $\gamma < \beta$ and $c\gamma = \beta$ where c is some positive integer then

$$E(\tilde{\tilde{S}}_{\alpha}) = S_{\alpha} + ka/m^{\gamma} + 0(m^{-2\gamma}) - 0(m^{-c\gamma}) . \qquad (4.16)$$

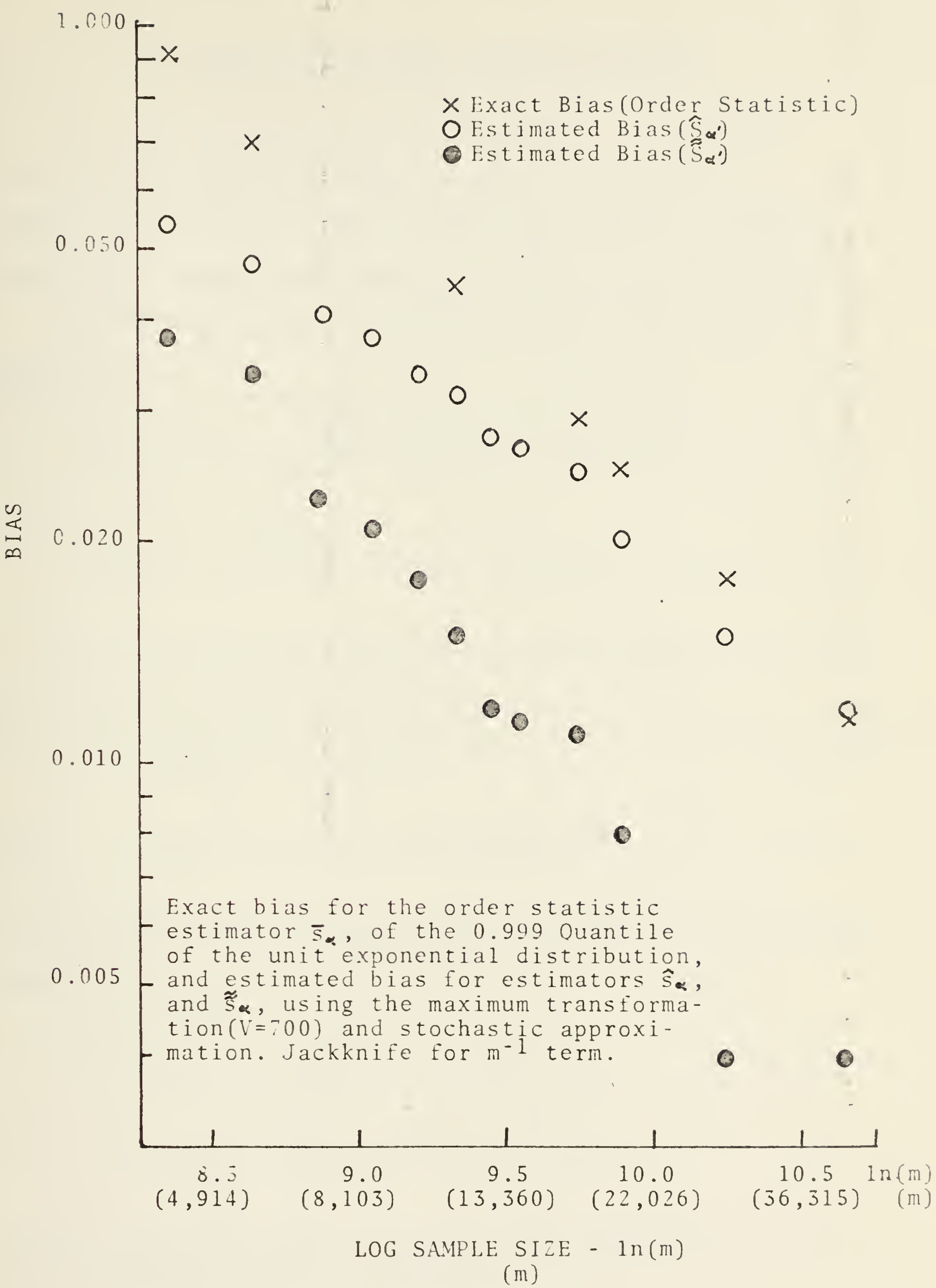
Consider the case where $\gamma = 0.5$ and the jackknife is applied for the m^{-1} term ($\beta = 1$). The resulting estimator will be biased and of the form

$$E(\tilde{\tilde{S}}_{\alpha}) = S_{\alpha} + .586\ am^{-.5} + 0(m^{-1.5}) . \qquad (4.17)$$

Thus there is a reduction in the magnitude of the $m^{-.5}$ term in the bias, but the leading term is still of the same order as before.

Casual inspection of the simulation results in Goodman, Lewis, and Robbins (1973) indicates that the bias is decreasing slower than m^{-1}. If the leading term is in fact $m^{-.5}$, the effect of the jackknife is precisely as in Equation 4.17. The bias results obtained by Goodman, Lewis, and Robbins (1973) are shown in Figure 1.

In the next sections the two types of quantile estimation are discussed in more detail.



× Exact Bias(Order Statistic)
○ Estimated Bias($\hat{S}_{\alpha'}$)
● Estimated Bias($\tilde{\hat{S}}_{\alpha'}$)
BIAS
1.000
0.050
0.020
0.010
0.005
Exact bias for the order statistic estimator $\bar{s}_\alpha$, of the 0.999 Quantile of the unit exponential distribution, and estimated bias for estimators $\hat{s}_\alpha$, and $\tilde{\hat{s}}_\alpha$, using the maximum transformation(V=700) and stochastic approximation. Jackknife for m^{-1} term.
8.5 (4,914)
9.0 (8,103)
9.5 (13,360)
10.0 (22,026)
10.5 (36,315)
ln(m) (m)
LOG SAMPLE SIZE - ln(m) (m)

FIGURE 1

V. QUANTILE ESTIMATE: ORDER STATISTIC

The order statistic estimate $\bar{S}_\alpha$ can be obtained by ordering the sample and using $S_{([\alpha m])}$ as the order statistic estimate $\bar{S}_\alpha$, where $[\alpha m]$ is the greatest integer less than or equal to αm and $S_{(i)}$ is the ith ordered value.

The distribution and density function of the ordered sample are:

$$F_{S(i)}(s) = \sum_{i=1}^{m} \binom{m}{k} [F(s)]^k [1 - F(s)]^{m-k} , \qquad (5.1)$$

$$f_{S(i)}(s) = \binom{m}{i} i [F(s)]^{i-1} [1-F(s)]^{m-i} f(s). \qquad (5.2)$$

For the uniform distribution (0,1) the mean and variance of the order statistic then become

$$E(S_{(i)}) = i/(m+1) , \qquad (5.3)$$

and

$$\mathrm{Var}(S_{(i)}) = i(m-i+1)/\{(m+1)(m+2)\}. \qquad (5.4)$$

If the sample size m is such that αm is equal to an integer, then $\bar{S}_\alpha = S_{([\alpha m])}$ and equations 5.3 and 5.4 become

$$E(\bar{S}_\alpha) = m/(m+1) = \alpha - [\alpha/(m+1)] \qquad (5.5)$$

and

$$\mathrm{Var}(\bar{S}_\alpha) = \alpha m(m-\alpha m+1)/\{(m+1)^2(m+2)\} . \qquad (5.6)$$

Results using equations 5.5 and 5.6 are tabulated in Table 1 for the uniform (0,1) 0.5 quantile.

Cox and Lewis (1966), using direct methods, arrived at the following results for the unit exponential distribution

$$E(\bar{S}_\alpha) = -\ln(1-\alpha) - \frac{1}{2m}\left(\frac{\alpha}{1-\alpha}\right) + \frac{1}{12m^2}\left[\frac{(1-\alpha)^2-1}{(1-\alpha)^2}\right] + 0(m^{-3}) \quad (5.7)$$

and $$\mathrm{Var}(\bar{S}_\alpha) = \frac{\alpha}{m(1-\alpha)} - \frac{1}{2m^2}\left[\frac{1}{(1-\alpha)^2} - 1\right] + 0(m^{-3}). \quad (5.8)$$

These equations were used to compute the results in Tables 2 through 6 for the unit exponential 0.5, 0.9, 0.95, 0.975, and 0.999 quantiles.

ORDER STATISTIC ESTIMATE: Uniform (0,1) Distribution
(Exact Results) .500 Quantile $S_{\alpha=.5}=.5000$

m	BIAS($\bar{S}_\alpha$)	VAR($\bar{S}_\alpha$)	STD DEV($\bar{S}_\alpha$)	MSE
8	-.055556	0.024691	0.157134	0.027777
16	-.029411	0.013840	0.117647	0.014705
24	-.020000	0.009600	0.097979	0.010000
32	-.015151	0.007346	0.085709	0.007575
40	-.012195	0.005949	0.077129	0.006097
48	-.010204	0.004997	0.070695	0.005102
56	-.008771	0.004309	0.065643	0.004385
64	-.007692	0.003786	0.061538	0.003846
72	-.006849	0.003377	0.058118	0.003424
80	-.006172	0.003048	0.055208	0.003086
88	-.005617	0.002777	0.052701	0.002808
96	-.005154	0.002550	0.050504	0.002577
104	-.004761	0.002358	0.048562	0.002380
112	-.004424	0.002192	0.046827	0.002212
120	-.004132	0.002049	0.045265	0.002066
128	-.003875	0.001922	0.043851	0.001937
136	-.003649	0.001811	0.042561	0.001824
144	-.003448	0.001712	0.041379	0.001724
152	-.003267	0.001623	0.040290	0.001633
160	-.003105	0.001543	0.039281	0.001552
168	-.002958	0.001470	0.038347	0.001479
176	-.002824	0.001404	0.037475	0.001412
184	-.002702	0.001344	0.036661	0.001351
192	-.002590	0.001288	0.035897	0.001295
200	-.002487	0.001238	0.035185	0.001244

TABLE 1

ORDER STATISTIC ESTIMATE: Unit Exponential Distribution
(Exact Results) .500 Quantile $S_{\alpha=.5}=0.693147$

m	BIAS($\bar{S}_\alpha$)	VAR($\bar{S}_\alpha$)	STD DEV($\bar{S}_\alpha$)	MSE
8	-.066406	0.101563	0.318689	0.105972
16	-.032227	0.056641	0.237993	0.057679
24	-.021267	0.039062	0.197642	0.039515
32	-.015869	0.029785	0.172584	0.030037
40	-.012656	0.024063	0.155121	0.024223
48	-.010525	0.020182	0.142064	0.020293
56	-.009008	0.017379	0.131829	0.017460
64	-.007874	0.015259	0.123526	0.015321
72	-.006993	0.013600	0.116617	0.013648
80	-.006289	0.012266	0.110750	0.012305
88	-.005714	0.011170	0.105688	0.011203
96	-.005235	0.010254	0.101262	0.010281
104	-.004831	0.009477	0.097348	0.009500
112	-.004484	0.008809	0.093856	0.008829
120	-.004184	0.008229	0.090715	0.008247
128	-.003972	0.007721	0.087869	0.007736
136	-.003690	0.007272	0.085275	0.007285
144	-.003484	0.006872	0.082898	0.006884
152	-.003300	0.006514	0.080710	0.006525
160	-.003135	0.006191	0.078685	0.006201
168	-.002985	0.005899	0.076806	0.005908
176	-.002849	0.005633	0.075056	0.005642
184	-.002725	0.005390	0.073420	0.005398
192	-.002611	0.005168	0.071886	0.005174
200	-.002506	0.004962	0.070445	0.004969

TABLE 2

ORDER STATISTIC ESTIMATE: Unit Exponential Distribution
(Exact Results) .900 Quantile $S_{\alpha=.9}=2.302585$

m	BIAS($\bar{S}_\alpha$)	VAR($\bar{S}_\alpha$)	STD DEV($\bar{S}_\alpha$)	MSE
56	-.082988	0.144930	0.380697	0.151817
112	-.040836	0.076411	0.276425	0.078079
168	-.027078	0.051818	0.227635	0.052551
224	-.020254	0.039192	0.197970	0.039602
280	-.016177	0.031511	0.177515	0.031773
336	-.013466	0.026347	0.162318	0.026529
392	-.011533	0.022637	0.150456	0.022770
448	-.010086	0.019843	0.140864	0.019944
504	-.008961	0.017662	0.132899	0.017743
560	-.008062	0.015914	0.126149	0.015979
616	-.007327	0.014480	0.120333	0.014534
672	-.006715	0.013283	0.115253	0.013328
728	-.006197	0.012269	0.110767	0.012308
784	-.005753	0.011399	0.106766	0.011432
840	-.005369	0.010644	0.103170	0.010673
896	-.005033	0.009983	0.099915	0.010008
952	-.004736	0.009399	0.096949	0.009422
1008	-.004472	0.008880	0.094233	0.008900
1064	-.004237	0.008415	0.091733	0.008433
1120	-.004024	0.007996	0.089422	0.008012
1176	-.003832	0.007617	0.087277	0.007632
1232	-.003658	0.007273	0.085279	0.007286
1288	-.003499	0.006958	0.083413	0.006970
1344	-.003353	0.006669	0.081664	0.006680
1400	-.003218	0.006403	0.080021	0.006414

TABLE 3

ORDER STATISTIC ESTIMATE: Unit Exponential Distribution
(Exact Results) .950 Quantile $S_{\alpha=.95}=2.995732$

m	BIAS($\bar{S}_\alpha$)	VAR($\bar{S}_\alpha$)	STD DEV($\bar{S}_\alpha$)	MSE
112	-.087472	0.153739	0.392095	0.161390
224	-.043073	0.080845	0.284333	0.082701
336	-.028568	0.054781	0.234052	0.055597
448	-.021371	0.041417	0.203511	0.041873
560	-.017070	0.033292	0.182462	0.033584
672	-.014211	0.027832	0.166829	0.028034
784	-.012171	0.023910	0.154629	0.024058
896	-.010644	0.020957	0.144765	0.021070
1008	-.009457	0.018653	0.136575	0.018742
1120	-.008509	0.016805	0.129635	0.016878
1232	-.007733	0.015291	0.123655	0.015350
1344	-.007087	0.014026	0.118433	0.014077
1456	-.006540	0.012955	0.113822	0.012998
1568	-.006072	0.012036	0.109710	0.012073
1680	-.005667	0.011239	0.106013	0.011271
1792	-.005312	0.010541	0.102667	0.010569
1904	-.004999	0.009924	0.099619	0.009949
2016	-.004720	0.009376	0.096827	0.009398
2128	-.004472	0.008885	0.094258	0.008905
2240	-.004248	0.008442	0.091882	0.008460
2352	-.004045	0.008042	0.089678	0.008059
2464	-.003861	0.007678	0.087625	0.007693
2576	-.003693	0.007346	0.085707	0.007359
2688	-.003539	0.007041	0.083910	0.007053
2800	-.003397	0.006760	0.082221	0.006772

TABLE 4

ORDER STATISTIC ESTIMATE: Unit Exponential Distribution
(Exact Results) .975 Quantile $S_{\alpha=.975}=3.688879$

m	BIAS($\bar{S}_\alpha$)	VAR($\bar{S}_\alpha$)	STD DEV($\bar{S}_\alpha$)	MSE
216	-.093134	0.163419	0.404252	0.172093
432	-.045853	0.085994	0.293247	0.088096
648	-.030410	0.058281	0.241415	0.059206
864	-.022748	0.044068	0.209924	0.044585
1080	-.018170	0.035426	0.188217	0.035756
1296	-.015126	0.029617	0.172095	0.029845
1512	-.012955	0.025444	0.159512	0.025612
1728	-.011329	0.022302	0.149338	0.022430
1944	-.010066	0.019850	0.140891	0.019951
2160	-.009056	0.017884	0.133732	0.017966
2376	-.008231	0.016273	0.127564	0.016340
2592	-.007543	0.014927	0.122177	0.014984
2808	-.006961	0.013787	0.117420	0.013836
3024	-.006463	0.012809	0.113179	0.012851
3240	-.006031	0.011961	0.109366	0.011997
3456	-.005654	0.011218	0.105914	0.011250
3672	-.005320	0.010562	0.102770	0.010590
3888	-.005024	0.009978	0.099890	0.010003
4104	-.004759	0.009455	0.097239	0.009478
4320	-.004521	0.008985	0.094789	0.009005
4536	-.004305	0.008559	0.092515	0.008578
4752	-.004109	0.008172	0.090397	0.008189
4968	-.003931	0.007818	0.088419	0.007833
5184	-.003767	0.007493	0.086564	0.007508
5400	-.003616	0.007195	0.084822	0.007208

TABLE 5

ORDER STATISTIC ESTIMATE: Unit Exponential Distribution
(Exact Results) .999 Quantile $S_{\alpha=.999} = 6.907755$

m	BIAS($\bar{S}_\alpha$)	VAR($\bar{S}_\alpha$)	STD DEV($\bar{S}_\alpha$)	MSE
5600	-.091854	0.162449	0.403050	0.170886
11200	-.045263	0.085210	0.291908	0.087259
16800	-.030027	0.057693	0.240193	0.058594
22400	-.022465	0.043602	0.208810	0.044106
28000	-.017946	0.035041	0.187192	0.035363
33600	-.014940	0.029289	0.171141	0.029512
39200	-.012797	0.025159	0.158617	0.025323
44800	-.011191	0.022050	0.148492	0.022175
50400	-.009944	0.019625	0.140088	0.019723
56000	-.008946	0.017680	0.132966	0.017760
61600	-.008131	0.016086	0.126830	0.016152
67200	-.007451	0.014755	0.121472	0.014811
72800	-.006877	0.013628	0.116740	0.013675
78400	-.006385	0.012661	0.112521	0.012702
84000	-.005958	0.011822	0.108729	0.011857
89600	-.005585	0.011087	0.105296	0.011118
95200	-.005256	0.010439	0.102169	0.010466
100800	-.004964	0.009862	0.099305	0.009886
106400	-.004702	0.009345	0.096669	0.009367
112000	-.004466	0.008880	0.094233	0.008900
117600	-.004253	0.008459	0.091971	0.008477
123200	-.004060	0.008076	0.089866	0.008092
128800	-.003883	0.007726	0.087898	0.007741
134400	-.003721	0.007405	0.086054	0.007419
140000	-.003572	0.007110	0.084322	0.007123

TABLE 6

VI. QUANTILE ESTIMATE: STOCHASTIC APPROXIMATION

The maximum transformation was used with the stochastic approximation in estimating extreme quantiles to alleviate the problem of slow convergence, as previously discussed. To implement the maximum transformation consider the sample S_i, $i=1,\ldots,m$, from the distribution $F(s)=\text{prob}(S=1)$ from which the extreme quantile S_α, $\alpha > 0.5$, is to be estimated. If the sample is partitioned into subsets of v values each (assuming $m'=m/v$) and the maximum value is taken from each of the m' subsets the result is a reduced sample $S'_1,\ldots,S'_{m'}$. The distribution of the random variable in the reduced sample $F_{max}(s)$ is then given by

$$F_{max}(s) = F^v(s) = \alpha^v = \alpha' \tag{6.1}$$

and consequently $S_\alpha' = S_\alpha$.

This result implies that if the transformation to the median $\alpha' = 0.5$ is used, and v is chosen such that $v = \ln(\alpha')/\ln(\alpha)$, then the stochastic approximation can be used to estimate the median quantile $S_\alpha' = .5$ of the reduced sample $S_1',\ldots, S_{m'}'$ which is equal to the extreme quantile S_α from the original sample.

The primary distribution used in the simulation was the unit exponential distribution, $F(s) = 1-e^{-s}$. To minimize the computing time of the program the maximum transformation was simulated by generating the reduced sample

directly. Since

$$F_{max}(s) = (1-e^{-s})^{v} = r \quad 0 \leq r \leq 1, \tag{6.2}$$

solving for s gives

$$s = -\ln(1-r^{1/v}). \tag{6.3}$$

Hence, m' pseudo-random numbers $r_1,\ldots,r_{m'}$ were generated and the reduced sample $S_1',\ldots,S'_{m'}$ was computed directly using equation 6.3 (i.e., $S_1' = -\ln(1-r_1{}^{1/v})$, etc.).[1] This procedure saved considerable computing time, if v is much greater than 10, by reducing the number of pseudo random numbers required from m down to $m' = m/v$. The values of the parameters used in the simulation are listed in the following table.

α	0.5	0.9	0.95	0.975	0.999
v	1	7	14	27	700
α'	0.5	0.47830	0.48768	0.50481	0.49641

The stochastic approximation was then applied to the reduced sample to obtain the estimate $\hat{S}_{\alpha'}(m')$ which is the value obtained on the m'th iteration using the following

[1]The random number generator, RANDOM, used was an adaptation of the generator reported in Lewis, P.A.W., Goodman, A.S., and Miller, J. M., "A Pseudo-Random Number Generator for the System/360," IBM Systems Journal, v.8, No. 2, 1969 The generator was modified by G. P. Learmonth to produce double precision pseudo-random numbers and a shuffling scheme implemented to improve the randomness of these numbers. Copies of the program may be obtained from G. P. Learmonth.

equation:

$$\hat{S}_{\alpha'}(i) = \hat{S}_{\alpha'}(i-1) - \frac{C}{i}\left[\frac{1-\operatorname{sgn}\{S_i' - \hat{S}_{\alpha'}(i-1)\}}{2} - \alpha'\right], \qquad (6.4)$$

where the function sgn(x) assumes the value 1 if $x > 0$ and 0 if $x \leq 0$, C is the inverse of the density $f\,(s_{\alpha'})$, and $\hat{S}_{\alpha'}(0)$ is an arbitrary starting value.

The initial values of C and $\hat{S}_{\alpha'}(0)$ were obtained using the first three sample values S_1', S_2', and S_3' from the reduced sample. These values were ordered to get $S_{(1)}'$, $S_{(2)}'$, and $S_{(3)}'$. The following approximations were then used for C and $\hat{S}_{\alpha'}(0)$.

$$C = \frac{8[(S_{(2)}' - S_{(1)}')(S_{(3)}' - S_{(2)}')]}{[(S_{(2)}' - S_{(1)}') + (S_{(3)}' - S_{(2)}')]} \quad \text{and } \hat{S}_{\alpha'}(0) = S_{(2)}'. \quad (6.5)$$

VII. RESULTS AND CONCLUSIONS

The results of the linear regression on the 0.999 quantile estimates for the unit exponential distribution without jackknife in Goodman, Lewis, and Robbins (1973) provided some evidence that the term of lower order than m^{-1} did exist. The results using equation 3.1 were

$$\text{bias} = 20.4m^{-.697} \tag{7.1}$$

and from equation 3.4 the results were

$$\text{bias} = 2.328m^{-.5} + 93.67m^{-1} . \tag{7.2}$$

From equation 7.2 it was apparent that the $m^{-.5}$ term was significant in the bias of the estimate even though the value $\gamma = 0.5$ was based on a conjecture only partially substantiated by the results in equation 7.1. The result that $\hat{\gamma} = 0.697$ can be partially accounted for by the fact that the m^{-1} term is still a significant factor at the sample values considered. The results of the linear regression on the jackknifed data in Goodman, Lewis, and Robbins (1973) did not provide any additional insight into the rate of convergence of the quantile estimate.

Although these results were not conclusive in establishing the exact order of the bias, they did tend to support the conjecture that the higher order term was $m^{-.5}$. It was therefore decided to extend m' out to a value of 200 and jackknife for a $m^{-.5}$ term.

The simulation was initially programmed to achieve an m-fold jackknife. However, results of the simulation were that the variance of the estimate tended to increase above a practical level. The variance of the jackknifed estimate for the 0.999 quantile of the unit exponential distribution was as much as 100 times higher than that of the stochastic approximation without the jackknife. These results are listed in Table 7 and may be compared with the final results of the 2-fold jackknife listed in Table 19 for the same quantile. It was thus concluded that without some kind of data transformation the m-fold jackknife was impractical. Consequently, the program was modified to investigate the 2-fold jackknife.

The variance reduction technique using control variables as discussed by Gaver (1969) was initially programmed into the simulation in an effort to reduce the run time required to achieve a reasonable value for the variance of the estimate. The control variable used was the number of sample values from the reduced sample that exceeded S_α divided by m' the total number of sample values. The result was that a reduction of approximately 20 per cent for the variance of the estimate of the expected value of the quantile estimate without the jackknife was achieved. However, the effect on the variance of the estimate of the expected value of the jackknifed estimate was negligible.

Converting the program from the m-fold to the 2-fold jackknife precluded the evaluation of Tukey's idea of

UNIT EXPONENTIAL .999 QUANTILE: $S_{\alpha=.999}=6.907755$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=700
m-FOLD JACKKNIFE FOR 1/(m**.5) TERM 30,000 REPLICATIONS
WITHOUT JACKKNIFE $\hat{S}_\alpha$ WITH JACKKNIFE $\tilde{\tilde{S}}_\alpha$

m	$E(\hat{S}_\alpha)$	$VAR(\hat{S}_\alpha)$	$E(\tilde{\tilde{S}}_\alpha)$	$VAR(\tilde{\tilde{S}}_\alpha)$
2800	6.96080589	0.56557834	6.87691307	9.86020279
5600	6.94756126	0.32208169	6.86427307	9.81397915
8400	6.93925476	0.23201281	6.86162376	9.20647907
11200	6.93278503	0.18805271	6.86303711	8.78336143
14000	6.92920494	0.16046524	6.88424015	8.48901558
16800	6.92753887	0.14253157	6.89339828	8.34396076
19600	6.92541790	0.12835014	6.89002037	8.20124626
22400	6.92293930	0.11829495	6.87098694	8.08727074
25200	6.90282214	0.11055136	6 87441063	8.00736809
28000	6.91970444	0.10370368	6.85559368	7.90486526
30800	6.91774845	0.09817553	6.85440636	7.94236660
33600	6.91738892	0.09331733	6.87031174	7.88332272
36400	6.91768169	0.08924949	6.88277817	7.82690239
39200	6.91680908	0.08552343	6.87920189	7.80613899
42000	6.91648483	0.08246911	6.89079285	7.79042625
44800	6.91583824	0.07976806	6.89094162	7.75193119
47600	6.91556931	0.07732958	6.89364243	7.74548054
50400	6.91496563	0.07515162	6.88003540	7.73441601

TABLE 7

estimating the variance of the quantile estimate using the variance of the pseudo-values. This assertion is based on the premise that the distribution of the pseudo-values tends to normality and that the pseudo-values are approximately uncorrelated. However, results from the data obtained by Goodman, Lewis, and Robbins (1973) for the coefficients of skewness and excess of the estimates obtained via a stochastic approximation tend to negate the assumption of a normal distribution. These results are plotted in Figures 2 and 3. If, as the theory of stochastic approximation asserts, the asymptotic distribution is normal, the convergence is very slow. It would therefore appear that using the pseudo-values to estimate the variance would not be fruitful in the present application of the jackknife without some kind of normalizing transformation. It may be that the basic problem is the correlation between pseudo-values, not the lack of normality; but this has not been investigated.

The final simulation was programmed to compute the stochastic approximation with maximum transformation with and without a 2-fold jackknife for the $m^{-.5}$ term. The results for the unit exponential 0.5, 0.9, 0.95, 0.975, and 0.999 quantile and the 0.5 quantile of the uniform (0,1) distribution are listed in Tables 8 through 19 with plots of the estimated bias versus sample size. The remaining evaluation is based on the results obtained for the unit exponential distribution.

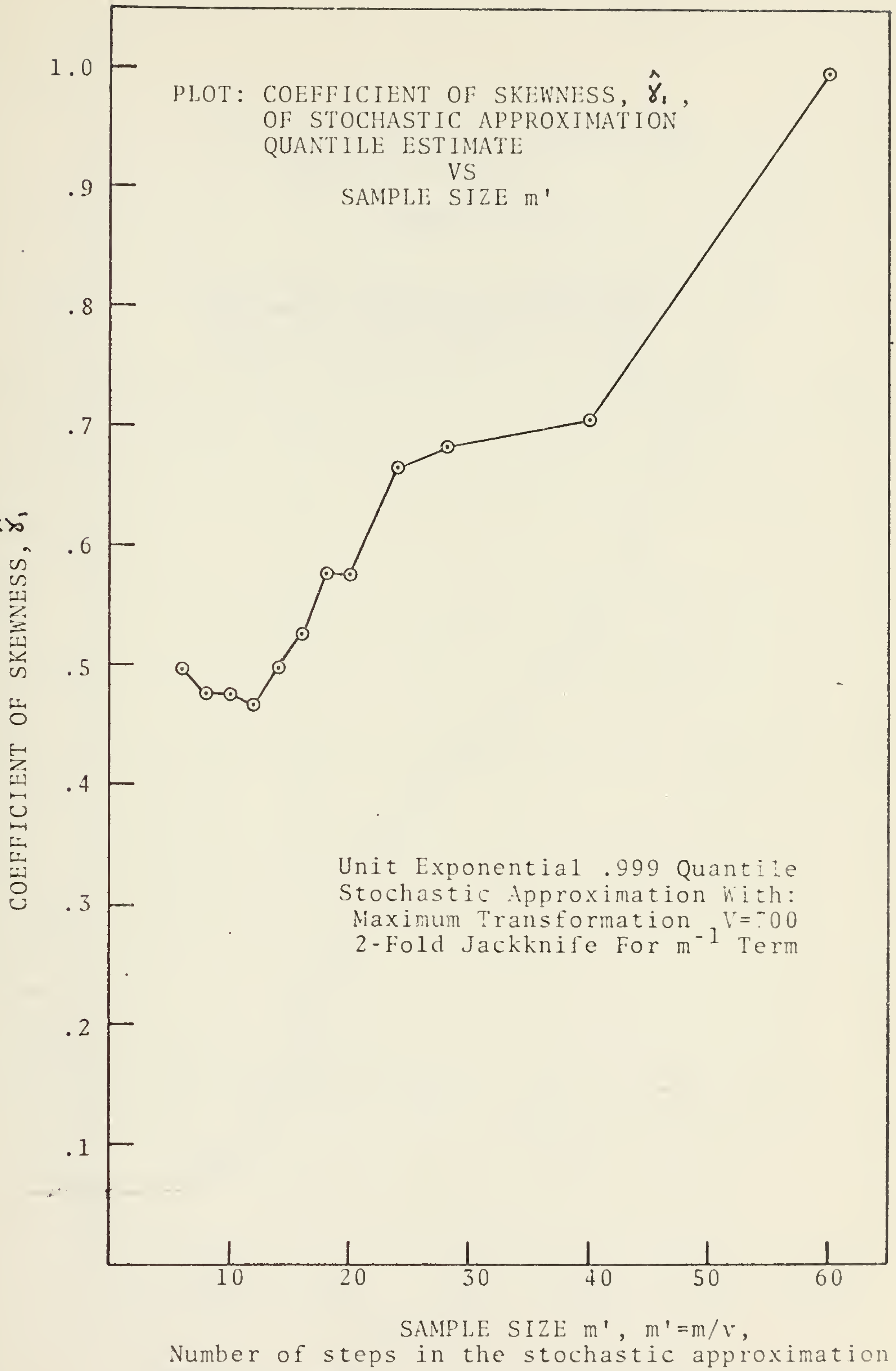

PLOT: COEFFICIENT OF SKEWNESS, $\hat{\gamma}_1$,
OF STOCHASTIC APPROXIMATION
QUANTILE ESTIMATE
VS
SAMPLE SIZE m'
Unit Exponential .999 Quantile
Stochastic Approximation With:
Maximum Transformation V=700
2-Fold Jackknife For m^{-1} Term
COEFFICIENT OF SKEWNESS, $\hat{\gamma}_1$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
10
20
30
40
50
60
SAMPLE SIZE m', m'=m/v,
Number of steps in the stochastic approximation

FIGURE 2

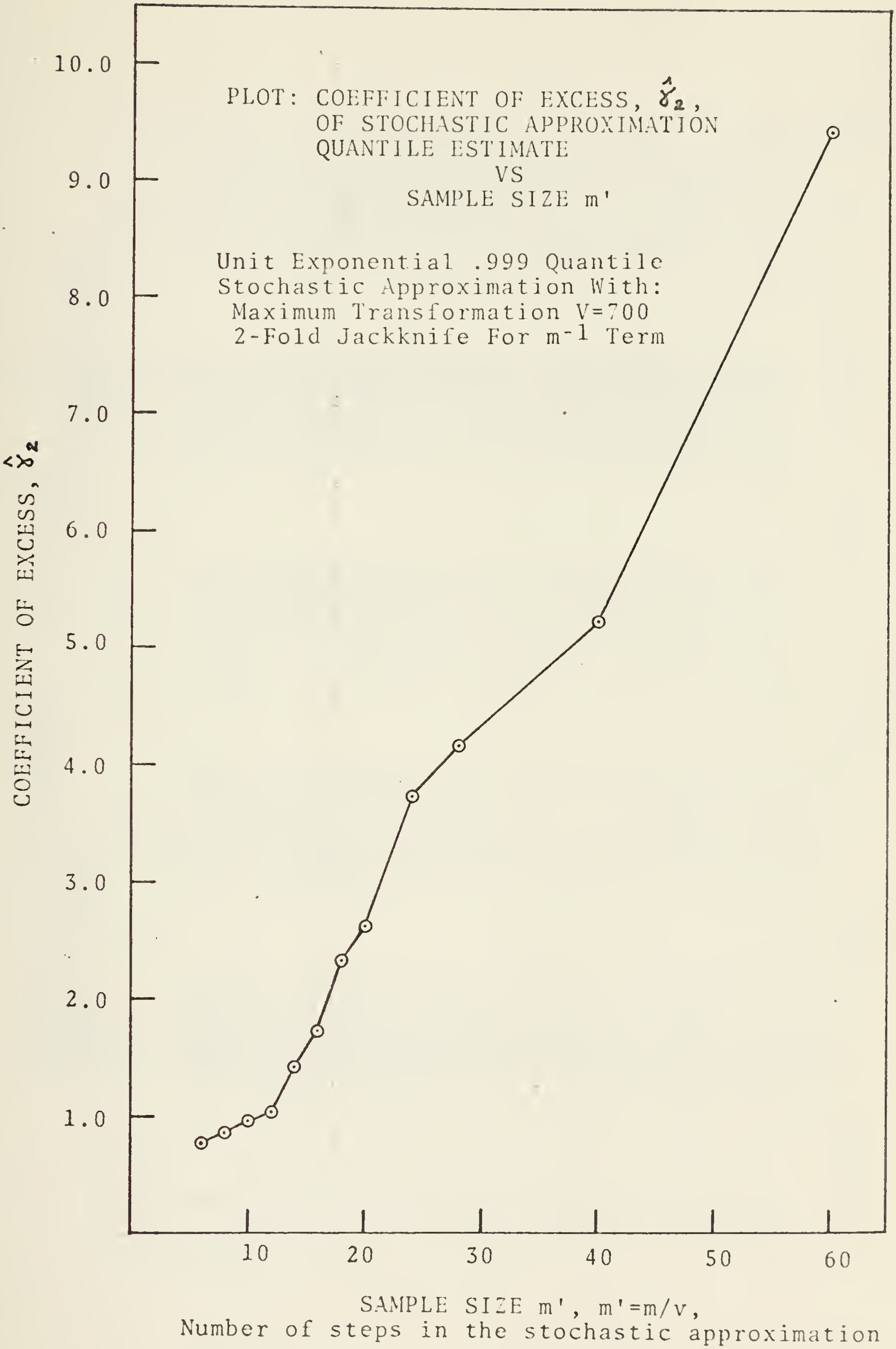

PLOT: COEFFICIENT OF EXCESS, $\hat{\gamma}_2$,
OF STOCHASTIC APPROXIMATION
QUANTILE ESTIMATE
VS
SAMPLE SIZE m'
Unit Exponential .999 Quantile
Stochastic Approximation With:
Maximum Transformation V=700
2-Fold Jackknife For m^{-1} Term
COEFFICIENT OF EXCESS, $\hat{\gamma}_2$
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
10
20
30
40
50
60
SAMPLE SIZE m', m'=m/v,
Number of steps in the stochastic approximation

FIGURE 3

UNIFORM (0,1) .500 QUANTILE: $S_{\alpha=.5}=.500$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=1
WITHOUT JACKKNIFE 90,000 REPLICATIONS

m	$E(\hat{S}_\alpha)$	$VAR(\hat{S}_\alpha)$	STD DEV$(\hat{S}_\alpha)$	BIAS$(\hat{S}_\alpha)$	MSE
8	0.50013346	0.02778017	0.16667384	0.00013346	0.02778018
16	0.49979350	0.01882480	0.13741353	-0.00020650	0.01882484
24	0.49985229	0.01522147	0.12337532	-0.00014771	0.01522149
32	0.49973197	0.01321831	0.11497090	-0.00026803	0.01321838
40	0.49990104	0.01193917	0.10926650	-0.00009896	0.01193918
48	0.49996413	0.01101040	0.10493045	-0.00003587	0.01101040
56	0.50003140	0.01031449	0.10156027	0.00003140	0.01031449
64	0.50011944	0.00975720	0.09877854	0.00011944	0.00975721
72	0.50007486	0.00932461	0.09656402	0.00007486	0.00932462
80	0.50012439	0.00895340	0.09462240	0.00012439	0.00895341
88	0.50013685	0.00863935	0.09294810	0.00013685	0.00863937
96	0.50008087	0.00836944	0.09148464	0.00008087	0.00836944
104	0.50002078	0.00813312	0.09018381	0.00002078	0.00813312
112	0.49996929	0.00792446	0.08901943	-0.00003071	0.00792446
120	0.50003177	0.00773894	0.08797124	0.00003177	0.00773894
128	0.50005983	0.00757016	0.08700666	0.00005983	0.00757017
136	0.50007587	0.00741774	0.08612630	0.00007587	0.00727622
144	0.50010531	0.00727621	0.08530070	0.00010531	0.00714940
152	0.50010753	0.00714939	0.08455406	0.00010753	0.00714940
160	0.50007481	0.00703685	0.08388593	0.00007481	0.00703685
168	0.50005843	0.00693354	0.08326788	0.00005843	0.00693354
176	0.50011234	0.00683583	0.08267907	0.00011234	0.00683584
184	0.50011426	0.00674498	0.08212782	0.00011426	0.00674499
192	0.50012293	0.00666101	0.08161501	0.00012293	0.00666103
200	0.50014634	0.00658183	0.08112231	0.00014634	0.00658185
	(.0010617)	(.0002060)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 8

UNIFORM (0,1) .500 QUANTILE: $S_{\alpha=.5}=.500$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=1
WITH JACKKNIFE $m^{-.5}$ TERM 90,000 REPLICATIONS

m	$E(\tilde{\tilde{S}}_\alpha)$	$VAR(\tilde{\tilde{S}}_\alpha)$	$STD\ DEV(\tilde{\tilde{S}}_\alpha)$	$BIAS(\tilde{\tilde{S}}_\alpha)$	MSE
8	0.49987548	0.09027701	0.30046132	-0.00012452	0.09027702
16	0.49941683	0.04885688	0.22103592	-0.00058317	0.04885722
24	0.49947293	0.03475839	0.18643602	-0.00052707	0.03475867
32	0.49943813	0.02805299	0.16749026	-0.00056187	0.02805331
40	0.49955422	0.02382034	0.15433839	-0.00044578	0.02382054
48	0.50017752	0.02111451	0.14530832	0.00017752	0.02111454
56	0.50011766	0.01919964	0.13856276	0.00011766	0.01919966
64	0.50031666	0.11761712	0.13272949	0.00031666	0.01761722
72	0.49994456	0.01632550	0.12777128	-0.00005544	0.01632551
80	0.49995233	0.01531031	0.12373483	-0.00004767	0.01531031
88	0.50009217	0.01446409	0.12026674	0.00009217	0.01446410
96	0.49996313	0.01385659	0.11771401	-0.00003687	0.01385660
104	0.49968261	0.01326132	0.11515780	-0.00031739	0.01326142
112	0.49960882	0.01274252	0.11288277	-0.00039118	0.01274268
120	0.49977387	0.01223974	0.11063335	-0.00022613	0.01223979
128	0.50028776	0.01185545	0.10888273	0.00028776	0.01185553
136	0.49983083	0.01144148	0.10696485	-0.00016917	0.01144151
144	0.49997062	0.01109622	0.10533859	-0.00002938	0.01109622
152	0.49983081	0.01083894	0.10411023	-0.00016919	0.01083897
160	0.49971931	0.01060304	0.10297261	-0.00028069	0.01060312
168	0.50003267	0.01040336	0.10199686	0.00003267	0.01040337
176	0.49979987	0.01012452	0.10062067	-0.00020013	0.01012456
184	0.50001454	0.00997350	0.09986741	0.00001454	0.00997350
192	0.50007281	0.00970022	0.09848969	0.00007281	0.00970023
200	0.50006914	0.00955754	0.09776267	0.00006914	0.00955755
	(.0009116)	(.0001756)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 9

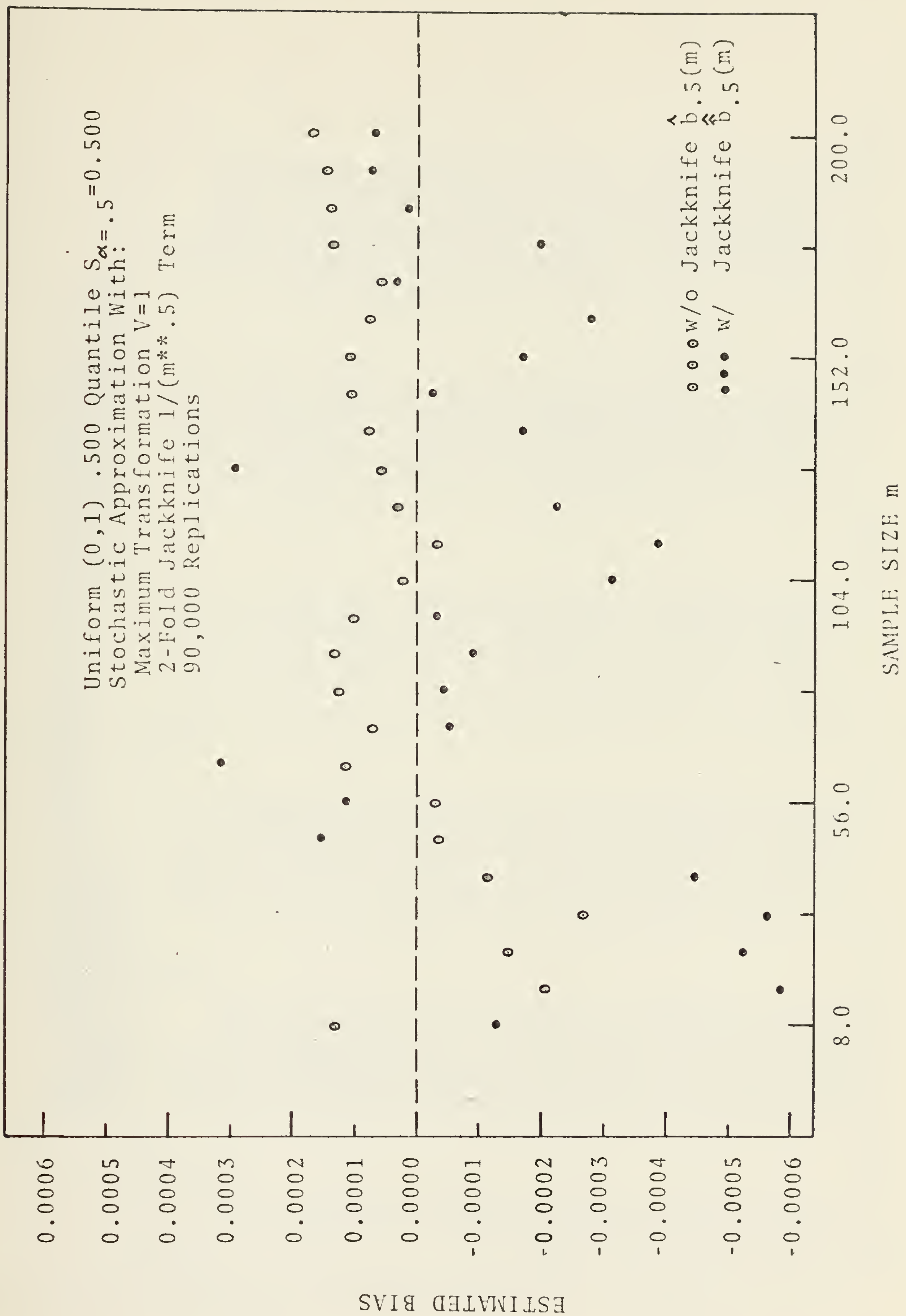

Uniform (0,1) .500 Quantile $S_\alpha = .5 = 0.500$
Stochastic Approximation With:
Maximum Transformation V=1
2-Fold Jackknife 1/(m**.5) Term
90,000 Replications
o o o w/o Jackknife $\hat{b}_{.5}(m)$
• • • w/ Jackknife $\hat{\hat{b}}_{.5}(m)$
0.0006
0.0005
0.0004
0.0003
0.0002
0.0001
0.0000
-0.0001
-0.0002
-0.0003
-0.0004
-0.0005
-0.0006
ESTIMATED BIAS
8.0
56.0
104.0
152.0
200.0
SAMPLE SIZE m

FIGURE 4

UNIT EXPONENTIAL .500 QUANTILE: $S_{\alpha=.5}=0.693147$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=1
WITHOUT JACKKNIFE 390,000 REPLICATIONS

m	$E(\hat{S}_\alpha)$	$VAR(\hat{S}_\alpha)$	STD DEV$(\hat{S}_\alpha)$	BIAS$(\hat{S}_\alpha)$	MSE
8	0.75499899	0.15716424	0.39643946	0.06185181	0.16098989
16	0.73173306	0.09656841	0.31075457	0.03858588	0.09805728
24	0.72085701	0.07467939	0.27327529	0.02770982	0.07544722
32	0.71406027	0.06301549	0.25102887	0.02091309	0.06345285
40	0.71009721	0.05572555	0.23606259	0.01695003	0.05601285
48	0.70722395	0.05063512	0.22502250	0.01407677	0.05083328
56	0.70491089	0.04687999	0.21651787	0.01176371	0.04701837
64	0.70319375	0.04395345	0.20965079	0.01004657	0.04405439
72	0.70161636	0.04164550	0.20407230	0.00846918	0.04171723
80	0.70041204	0.03975181	0.19937856	0.00726486	0.03980459
88	0.69945583	0.03814681	0.19531209	0.00630865	0.03818661
96	0.69855354	0.03675884	0.19172595	0.00540636	0.03678807
104	0.69776232	0.03558426	0.18863792	0.00461514	0.03560556
112	0.69703293	0.03455651	0.18589381	0.00388575	0.03457161
120	0.69650713	0.03363647	0.18340247	0.00335995	0.03364776
128	0.69605285	0.03281906	0.18116033	0.00290567	0.03282751
136	0.69561156	0.03208667	0.17912752	0.00246438	0.03209274
144	0.69517780	0.03141319	0.17723766	0.00203062	0.03141731
152	0.69480387	0.03080795	0.17552193	0.00165668	0.03081069
160	0.69441697	0.03027182	0.17398800	0.00126978	0.03027344
168	0.69410309	0.02976688	0.17253082	0.00095591	0.02976780
176	0.69384917	0.02930121	0.17117596	0.00070199	0.02930170
184	0.69360754	0.02886438	0.16989520	0.00046036	0.02886459
192	0.69329923	0.02845655	0.16869068	0.00015205	0.02845657
200	0.69305887	0.02808854	0.16759637	-0.00008831	0.02808855
	(.0021280)	(.0021989)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 10

UNIT EXPONENTIAL .500 QUANTILE: $S_{\alpha=.5}=0.693147$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=1
WITH JACKKNIFE $m^{-.5}$ TERM 390,000 REPLICATIONS

m	$E(\tilde{\tilde{S}}_\alpha)$	$VAR(\tilde{\tilde{S}}_\alpha)$	STD DEV$(\tilde{\tilde{S}}_\alpha)$	BIAS$(\tilde{\tilde{S}}_\alpha)$	MSE
8	0.70838103	0.81325615	0.90180716	0.01523385	0.81348822
16	0.67680562	0.37395581	0.61151926	-0.01634156	0.37422285
24	0.67337416	0.24003642	0.48993511	-0.01977302	0.24042739
32	0.67246760	0.17836972	0.42233839	-0.02067958	0.17879736
40	0.67372998	0.14282716	0.37792481	-0.01941720	0.14320419
48	0.67458317	0.11954477	0.34575247	-0.01856401	0.11988940
56	0.67535715	0.10377238	0.32213720	-0.01779003	0.10408886
64	0.67617351	0.09229660	0.30380356	-0.01697367	0.09258471
72	0.67655482	0.08348400	0.28893598	-0.01659236	0.08375930
80	0.67699692	0.07654253	0.27666320	-0.01615026	0.07680336
88	0.67754868	0.07083882	0.26615564	-0.01559850	0.07108214
96	0.67783396	0.06615951	0.25721492	-0.01531322	0.06639401
104	0.67817964	0.06226514	0.24952984	-0.01496754	0.06248917
112	0.67840920	0.05895405	0.24280455	-0.01473798	0.05917126
120	0.67876502	0.05605005	0.23674891	-0.01438217	0.05625689
128	0.67914575	0.05356482	0.23144074	-0.01400143	0.05376086
136	0.67946416	0.05138326	0.22667876	-0.01368302	0.05157048
144	0.67963587	0.04941020	0.22228405	-0.01351131	0.04959275
152	0.67987288	0.04765963	0.21831085	-0.01327430	0.04783583
160	0.68004953	0.04611735	0.21474951	-0.01309765	0.04628890
168	0.68025583	0.04468500	0.21138826	-0.01289135	0.04485118
176	0.68044236	0.04341023	0.20835122	-0.01270482	0.04357164
184	0.68062434	0.04223167	0.20550346	-0.01252284	0.04328849
192	0.68072630	0.04116212	0.20288451	-0.01242088	0.04131640
200	0.68083968	0.04020223	0.20050495	-0.01230750	0.04035371
	(.0026830)	(.0019789)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 11

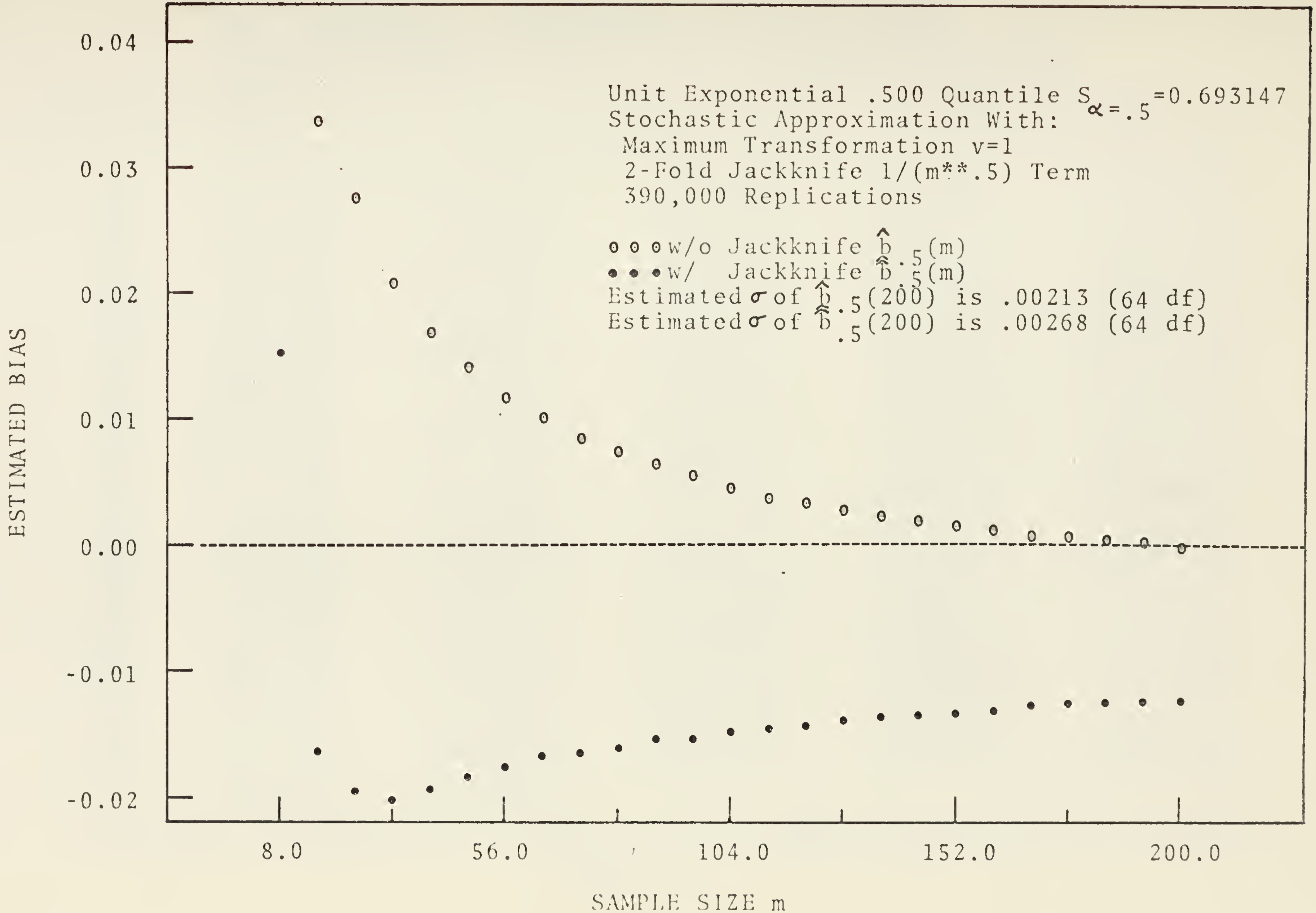

0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
ESTIMATED BIAS
8.0
56.0
104.0
152.0
200.0
SAMPLE SIZE m
Unit Exponential .500 Quantile $S_{\alpha=.5}$=0.693147
Stochastic Approximation With:
Maximum Transformation v=1
2-Fold Jackknife 1/(m**.5) Term
390,000 Replications
o o o w/o Jackknife $\hat{b}_{.5}(m)$
• • • w/ Jackknife $\hat{\hat{b}}_{.5}(m)$
Estimated σ of $\hat{b}_{.5}(200)$ is .00213 (64 df)
Estimated σ of $\hat{\hat{b}}_{.5}(200)$ is .00268 (64 df)

FIGURE 5

UNIT EXPONENTIAL .900 QUANTILE: $S_{\alpha=.9}=2.302585$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=7
WITHOUT JACKKNIFE 60,000 REPLICATIONS

m	$E(\hat{S}_\alpha)$	$VAR(\hat{S}_\alpha)$	STD DEV$(\hat{S}_\alpha)$	BIAS$(\hat{S}_\alpha)$	MSE
56	2.35334573	0.27708131	0.52638514	0.05076064	0.27965795
112	2.33621713	0.16412155	0.40511918	0.03363204	0.16525266
168	2.32953008	0.12340218	0.35128647	0.02694499	0.12412821
224	2.23403190	0.10285876	0.32071601	0.02144680	0.10331872
280	2.32204600	0.09049412	0.30082241	0.01946091	0.09087285
336	2.31955826	0.08131319	0.28515467	0.01697316	0.08160127
392	2.31797406	0.07499856	0.27385865	0.01538896	0.07523538
448	2.31704119	0.06978725	0.26417277	0.01445610	0.06999623
504	2.31566088	0.06587131	0.25665407	0.01307578	0.06604229
560	2.31487321	0.06259595	0.25019183	0.01228812	0.06274695
616	2.31389844	0.05976529	0.24446941	0.01131335	0.05989328
672	2.31317587	0.05744775	0.23968260	0.01059078	0.05755991
728	2.31246029	0.05544547	0.23546862	0.00987519	0.05554299
784	2.31163111	0.05360881	0.23153577	0.00904602	0.05369064
840	2.31167520	0.05205582	0.22815746	0.00909011	0.05213845
896	2.31122329	0.05061229	0.22497175	0.00863820	0.05068691
952	2.31094869	0.04935169	0.22215240	0.00836359	0.04942164
1008	2.31075249	0.04815172	0.21943500	0.00816740	0.04821842
1064	2.31048714	0.04710006	0.21702548	0.00790205	0.04716250
1120	2.31010097	0.04620159	0.21494555	0.00751588	0.04625808
1176	2.30989297	0.04534414	0.21294164	0.00730788	0.04539755
1232	2.30990433	0.04455133	0.21107185	0.00731924	0.04460490
1288	2.30984538	0.04380640	0.20929978	0.00726029	0.04385911
1344	2.30949591	0.04314220	0.20770701	0.00691082	0.04318996
1400	2.30937113	0.04253540	0.20624112	0.00678604	0.04258145
	(.0017268)	(.0017479)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 12

UNIT EXPONENTIAL .900 QUANTILE: $S_{\alpha=.9}=2.302585$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=7
WITH JACKKNIFE $m^{-.5}$ TERM 60,000 REPLICATIONS

m	$E(\tilde{\tilde{S}}_\alpha)$	$VAR(\tilde{\tilde{S}}_\alpha)$	STD DEV$(\tilde{\tilde{S}}_\alpha)$	BIAS$(\tilde{\tilde{S}}_\alpha)$	MSE
56	2.32357068	1.55892414	1.24856884	0.02098558	1.55936454
112	2.29426846	0.68701397	0.82886306	-0.00831664	0.68708314
168	2.29435988	0.42529579	0.65214706	-0.00822521	0.42536344
224	2.29247681	0.31026663	0.55701582	-0.01010829	0.31036880
280	2.29540084	0.24765084	0.49764530	-0.00718426	0.24770246
336	2.29605071	0.20435951	0.45206140	-0.00653439	0.20440221
392	2.29629371	0.17669252	0.42034809	-0.00629138	0.17673210
448	2.29716257	0.15575782	0.39466165	-0.00542252	0.15578723
504	2.29714610	0.14026384	0.37541814	-0.00543899	0.14029342
560	2.29701656	0.12795706	0.35771086	-0.00556853	0.12798807
616	2.29682925	0.11774867	0.34314526	-0.00575584	0.11778180
672	2.29685372	0.10961244	0.33107770	-0.00573137	0.10964529
728	2.29679909	0.10252875	0.32020115	-0.00578600	0.10256223
784	2.29666156	0.09648949	0.31062757	-0.00592354	0.09652458
840	2.29750935	0.09138113	0.30229312	-0.00507574	0.09140689
896	2.29745411	0.08720441	0.29530393	-0.00513098	0.08723074
952	2.29774336	0.08337399	0.28874554	-0.00484173	0.08339743
1008	2.29809689	0.07971103	0.28233141	-0.00448820	0.07973117
1064	2.29832914	0.07668680	0.27692381	-0.00425595	0.07670491
1120	2.29819161	0.07394834	0.27193445	-0.00440348	0.07396774
1176	2.29850621	0.07136093	0.26713466	-0.00407888	0.07137757
1232	2.29888765	0.06917864	0.26301833	-0.00369744	0.06919231
1288	2.29934574	0.06727161	0.25936771	-0.00323935	0.6728210
1344	2.29922306	0.06538840	0.25571156	-0.00336203	0.06539971
1400	2.29925719	0.06370420	0.25239692	-0.00332790	0.06371528
	(.0028928)	(.0016798)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 13

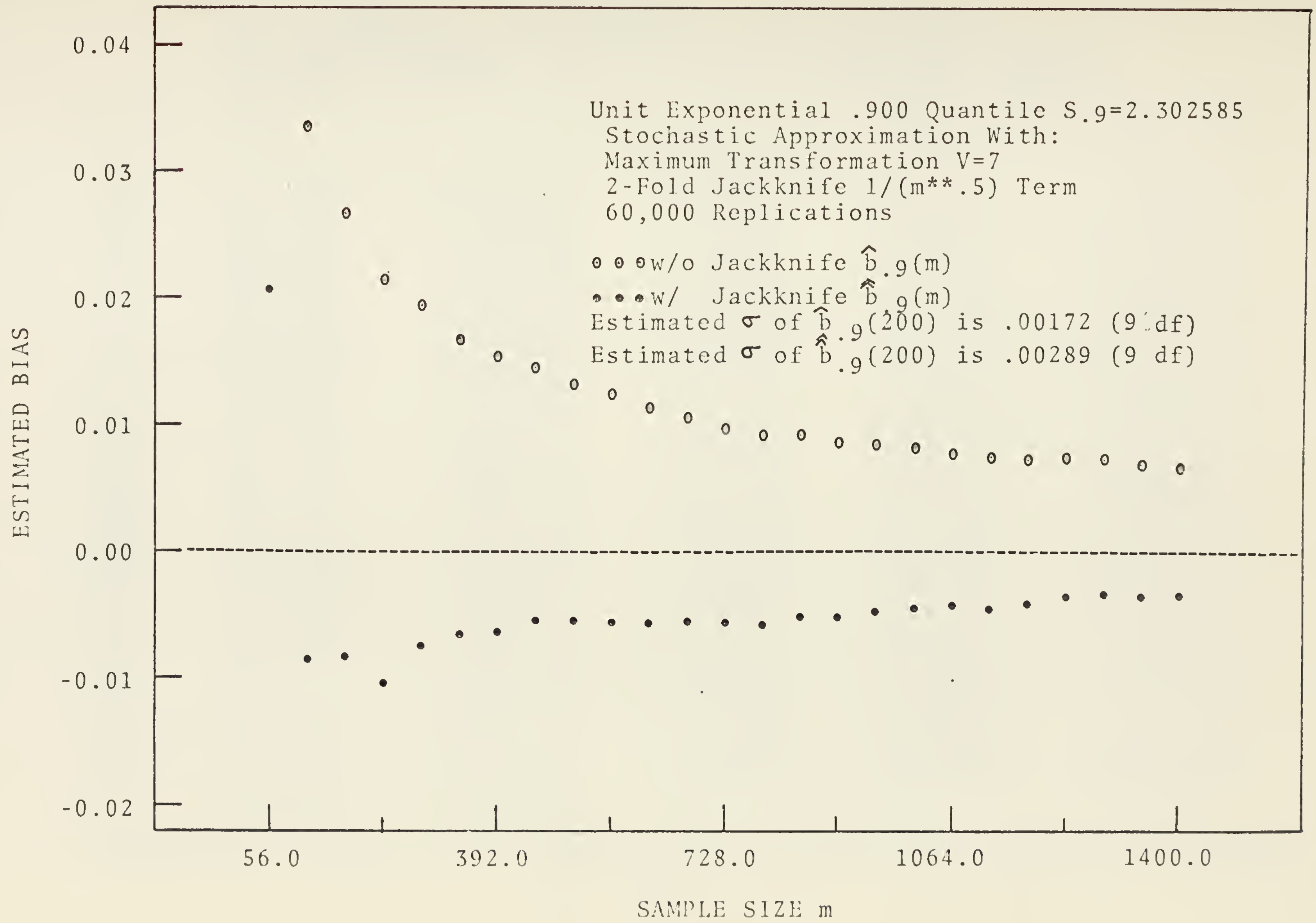

Unit Exponential .900 Quantile $S_{.9}$=2.302585
Stochastic Approximation With:
Maximum Transformation V=7
2-Fold Jackknife 1/(m**.5) Term
60,000 Replications
w/o Jackknife $\hat{b}_{.9}(m)$
w/ Jackknife $\hat{\hat{b}}_{.9}(m)$
Estimated σ of $\hat{b}_{.9}(200)$ is .00172 (9 df)
Estimated σ of $\hat{\hat{b}}_{.9}(200)$ is .00289 (9 df)
0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
ESTIMATED BIAS
56.0
392.0
728.0
1064.0
1400.0
SAMPLE SIZE m

FIGURE 6

UNIT EXPONENTIAL .950 QUANTILE: $S_{\alpha=.95}=2.995732$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=14
WITHOUT JACKKNIFE 60,000 REPLICATIONS

m	$E(\hat{S}_\alpha)$	$VAR(\hat{S}_\alpha)$	STD DEV$(\hat{S}_\alpha)$	BIAS$(\hat{S}_\alpha)$	MSE
112	3.04290076	0.29589152	0.54395911	0.04716848	0.29811638
224	3.02660403	0.17460237	0.41785448	0.03087175	0.17555543
336	3.02026014	0.13149184	0.36261804	0.02452787	0.13209346
448	3.01469342	0.10939786	0.33075347	0.01896115	0.10975738
560	3.01266321	0.09610400	0.31000646	0.01693094	0.09639066
672	3.01019878	0.08648222	0.29407860	0.01446651	0.08669150
784	3.00882383	0.07970535	0.28232136	0.01309155	0.07987674
896	3.00796144	0.07412211	0.27225376	0.01222917	0.07427166
1008	3.00667303	0.06993801	0.26445796	0.01094075	0.07005771
1120	3.00601422	0.06644512	0.25776951	0.01028194	0.06655084
1232	3.00515933	0.06342191	0.25183706	0.00942706	0.06351078
1344	3.00444901	0.06093264	0.24684538	0.00871674	0.06100862
1456	3.00373829	0.05876538	0.24241572	0.00800602	0.05882948
1568	3.00295122	0.05683754	0.23840626	0.00721895	0.05688966
1680	3.00298026	0.05515948	0.23486055	0.00724798	0.05521201
1792	3.00248916	0.05361140	0.23154135	0.00675689	0.05365705
1904	3.00222622	0.05225121	0.22858523	0.00649394	0.05229338
2016	3.00209626	0.05098597	0.22580073	0.00636399	0.05102647
2128	3.00184533	0.04985275	0.22327730	0.00611305	0.04989012
2240	3.00151422	0.04889704	0.22112676	0.00578194	0.04893048
2352	3.00131934	0.04798878	0.21906341	0.00558706	0.04801999
2464	3.00135219	0.04712735	0.21708835	0.00561991	0.04715894
2576	3.00128304	0.04633352	0.21525223	0.00555077	0.04636433
2688	3.00092142	0.04561943	0.21358705	0.00518915	0.04564635
2800	3.00082830	0.04497008	0.21206150	0.00509603	0.04499605
	(.0018011)	(.0018016)			

Quantities in brackets are estimates of the standard deviations of the estimates.

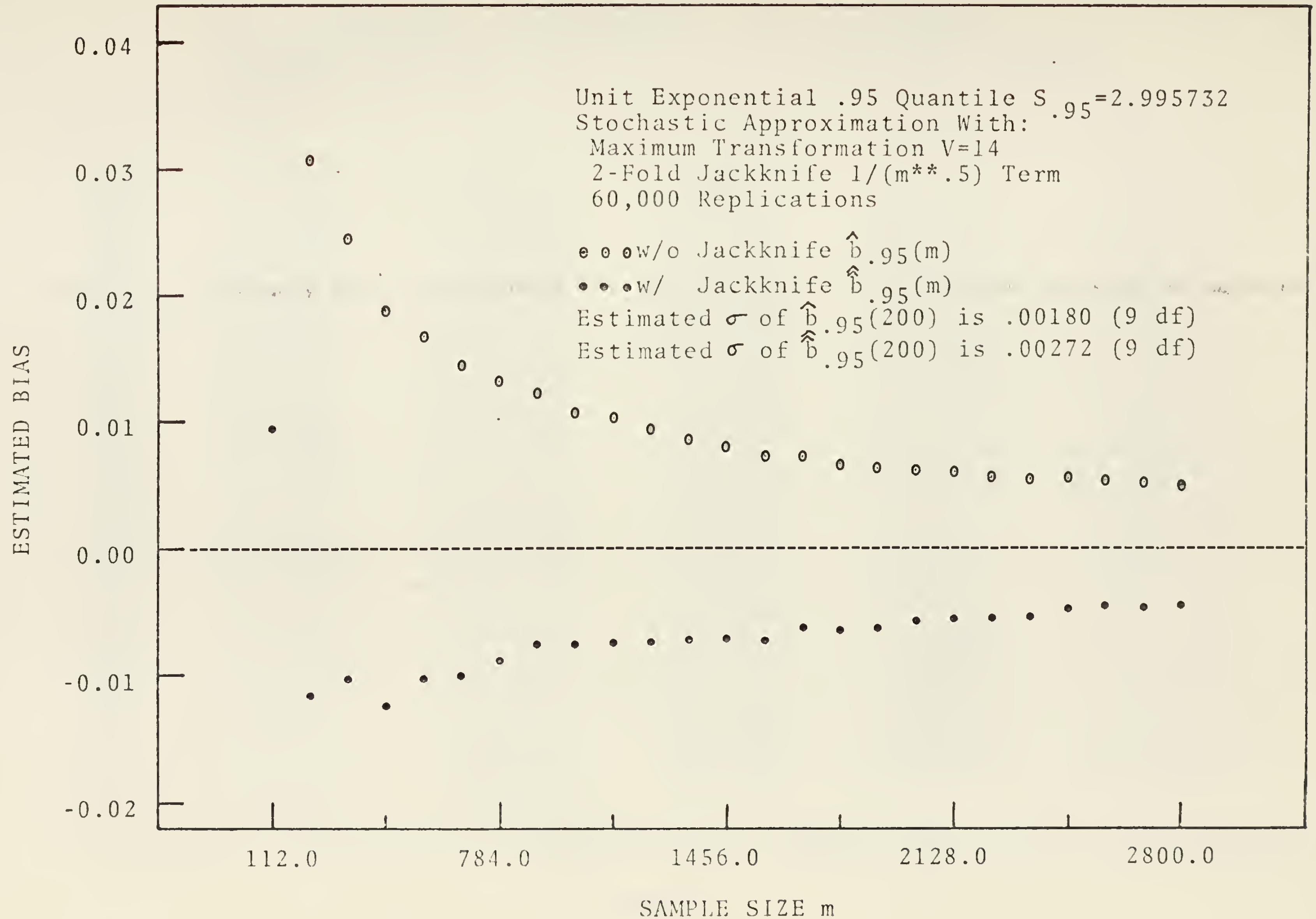
TABLE 14

UNIT EXPONENTIAL .950 QUANTILE: $S_{\alpha=.95}=2.995732$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=14
WITH JACKKNIFE $m^{-.5}$ TERM 60,000 REPLICATIONS

m	$E(\tilde{\tilde{S}}_\alpha)$	$VAR(\tilde{\tilde{S}})$	STD DEV$(\tilde{\tilde{S}}_\alpha)$	BIAS$(\tilde{\tilde{S}}_\alpha)$	MSE
112	3.00513860	1.67983655	1.29608509	0.00940632	1.67992503
224	2.98408233	0.72978790	0.85427624	-0.01164995	0.72992362
336	2.98570367	0.45556538	0.67495583	-0.01002860	0.45566595
448	2.98339273	0.33077380	0.57512938	-0.01233954	0.33092606
560	2.98540143	0.26257691	0.51242259	-0.01033084	0.26268364
672	2.98565859	0.21833186	0.46725995	-0.01007369	0.21843334
784	2.98686801	0.18870102	0.43439731	-0.00886426	0.18877960
896	2.98798268	0.16607937	0.40752837	-0.00774959	0.16613943
1008	2.98797476	0.14956206	0.38673254	-0.00775752	0.14962224
1120	2.98829428	0.13624867	0.36911877	-0.00743799	0.13630399
1232	2.98841258	0.12551179	0.35427643	-0.00731970	0.12556537
1344	2.98869075	0.11695897	0.34199264	-0.00704153	0.11700855
1456	2.98872175	0.10927620	0.33056951	-0.00701053	0.10932535
1568	2.98870087	0.10286257	0.32072195	-0.00703140	0.10291201
1680	2.98966699	0.09731148	0.31194788	-0.00606528	0.09734827
1792	2.98936173	0.09273933	0.30453132	-0.00637054	0.09277991
1904	2.98967195	0.08861418	0.29768134	-0.00606033	0.08865091
2016	2.99008657	0.08484376	0.29127952	-0.00564571	0.08487564
2128	2.99019770	0.08162989	0.28570945	-0.00553458	0.08166052
2240	2.99019044	0.07877703	0.28067246	-0.00554183	0.07880774
2352	2.99050162	0.07609196	0.27584771	-0.00523066	0.07611932
2464	2.99091045	0.07365952	0.27140287	-0.00482182	0.07368277
2576	2.99132140	0.07148606	0.26736877	-0.00441087	0.07150552
2688	2.99116238	0.06949821	0.26362514	-0.00456990	0.06951910
2800	2.99137196	0.06770790	0.26020742	-0.00436031	0.06772691
	(.0027238)	(.0016162)			

Quantities in bracket are estimates of the standard deviations of the estimates.

TABLE 15


Unit Exponential .95 Quantile $S_{.95}$=2.995732
Stochastic Approximation With:
Maximum Transformation V=14
2-Fold Jackknife 1/(m**.5) Term
60,000 Replications
w/o Jackknife $\hat{b}_{.95}(m)$
w/ Jackknife $\hat{\hat{b}}_{.95}(m)$
Estimated σ of $\hat{b}_{.95}(200)$ is .00180 (9 df)
Estimated σ of $\hat{\hat{b}}_{.95}(200)$ is .00272 (9 df)
ESTIMATED BIAS
0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
112.0
784.0
1456.0
2128.0
2800.0
SAMPLE SIZE m
FIGURE 7

UNIT EXPONENTIAL .975 QUANTILE: $S_{\alpha=.975}=3.688879$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=27
WITHOUT JACKKNIFE 60,000 REPLICATIONS

m	$E(\hat{S}_\alpha)$	$VAR(\hat{S}_\alpha)$	STD DEV$(\hat{S}_\alpha)$	BIAS$(\hat{S}_\alpha)$	MSE
216	3.73560574	0.31137524	0.55801008	0.04672629	0.31355859
432	3.71716770	0.18391621	0.42885454	0.02828825	0.18471644
648	3.71036216	0.13800887	0.37149545	0.02148271	0.13847038
864	3.70469934	0.11477009	0.33877734	0.01581989	0.11502036
1080	3.70247632	0.10060958	0.31719014	0.01359687	0.10079446
1296	3.69981728	0.09038285	0.30063740	0.01093782	0.09050248
1512	3.69841580	0.08327572	0.28857533	0.00953634	0.08336666
1728	3.69761747	0.07746197	0.27831991	0.00873801	0.07753833
1944	3.69629157	0.07305508	0.27028703	0.00741212	0.07311002
2160	3.69563424	0.06943162	0.26349880	0.00675479	0.06947724
2376	3.69486345	0.06624338	0.25737789	0.00598399	0.06627919
2592	3.69415805	0.06363970	0.25226911	0.00527860	0.06366757
2808	3.69344580	0.06133516	0.24765935	0.00456635	0.06135601
3024	3.69263144	0.05930498	0.24352613	0.00375198	0.05931905
3240	3.60272159	0.05756753	0.23993234	0.00384214	0.05758229
3456	3.69219909	0.05587660	0.23638232	0.00331964	0.05588762
3672	3.69189231	0.05445524	0.23335646	0.00301286	0.05446431
3888	3.69172530	0.05312111	0.23048017	0.00284584	0.05312921
4104	3.69154532	0.05195595	0.22793848	0.00266586	0.05196306
4320	3.69116388	0.05094470	0.22570932	0.00228442	0.05094991
4536	3.69098438	0.04997890	0.22355961	0.00210493	0.04998333
4752	3.69104646	0.04909047	0.22156369	0.00216701	0.04909517
4968	3.69096124	0.04824353	0.21964411	0.00208179	0.04824787
5184	3.69061148	0.04748395	0.21790813	0.00173203	0.04748695
5400	3.69053350	0.04680234	0.21633848	0.00165405	0.04680507
	(.0017089)	(.0018141)			

Quantities in brackets are estimates of the standard deviations of the estimates.

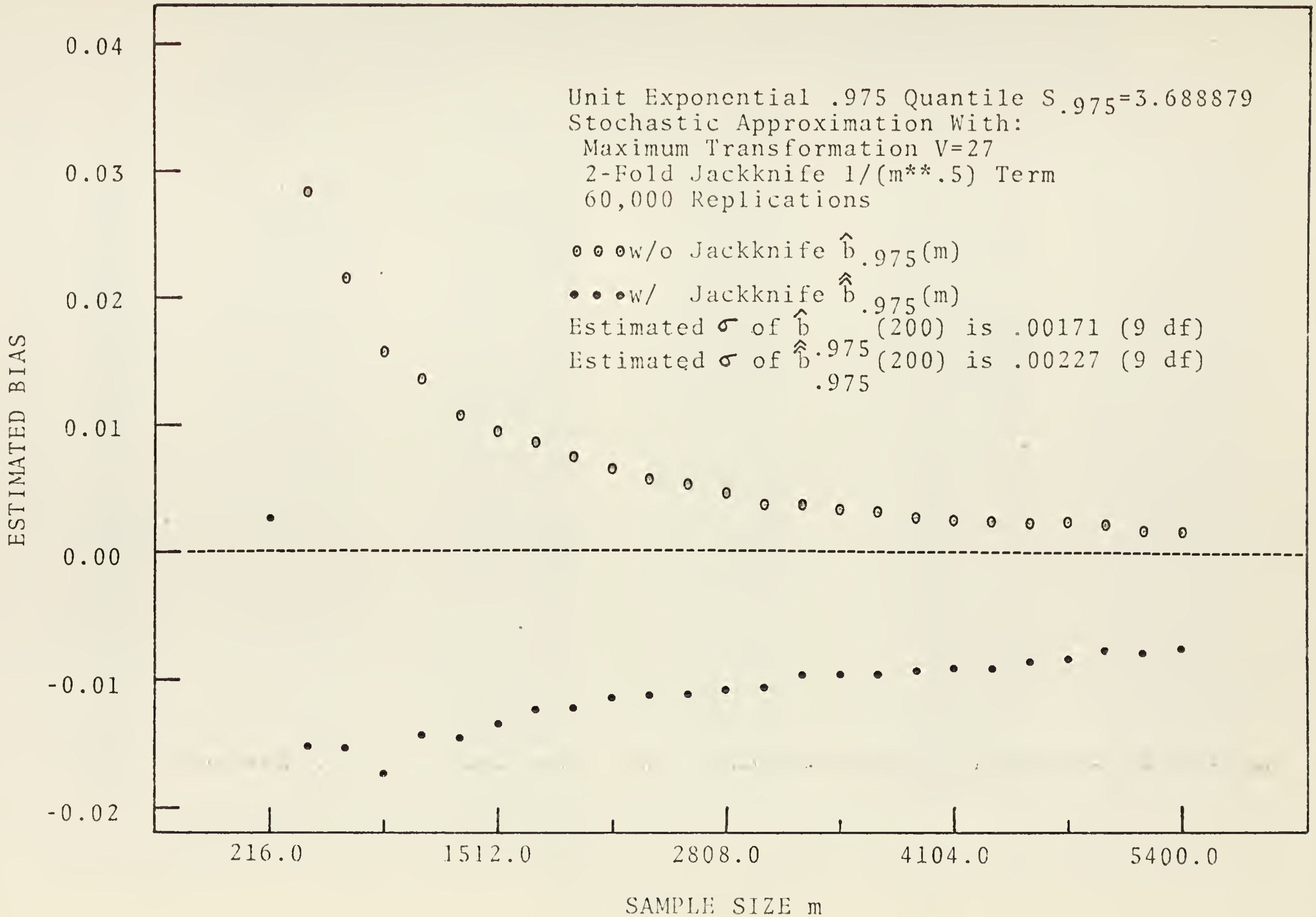
TABLE 16

UNIT EXPONENTIAL .975 QUANTILE: $S_{\alpha=.975}=3.688879$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=27
WITH JACKKNIFE $m^{-.5}$ TERM 60,000 REPLICATIONS

m	$E(\tilde{\tilde{S}}_\alpha)$	$VAR(\tilde{\tilde{S}}_\alpha)$	$STD\ DEV(\tilde{\tilde{S}}_\alpha)$	$BIAS(\tilde{\tilde{S}}_\alpha)$	MSE
216	3.69170977	1.75177365	1.32354586	0.00283032	1.75178166
432	3.67371968	0.77291649	0.87915669	-0.01515978	0.77314631
648	3.67359960	0.48447223	0.69604039	-0.01527986	0.48470570
864	3.67164402	0.35181157	0.59313706	-0.01723543	0.35210863
1080	3.67447487	0.27876709	0.52798399	-0.01440458	0.27897458
1296	3.67401044	0.23019262	0.47978393	-0.01486902	0.23041370
1512	3.67533675	0.19843201	0.44545707	-0.01354271	0.19861541
1728	3.67649354	0.17474289	0.41802260	-0.01238591	0.17489630
1944	3.67677173	0.15717696	0.39645550	-0.01210772	0.15732356
2160	3.67757451	0.14354115	0.37886825	-0.01130494	0.14366895
2376	3.67772255	0.13179241	0.36303224	-0.01115690	0.13191688
2592	3.67787373	0.12272595	0.35032263	-0.01100572	0.12284707
2808	3.67789076	0.11466995	0.33862952	-0.01098870	0.11479070
3024	3.67798987	0.10800948	0.32864796	-0.01088958	0.10812806
3240	3.67925787	0.10210918	0.31954528	-0.00962158	0.10220176
3456	3.67902361	0.09689151	0.31127401	-0.00985585	0.09698865
3672	3.67914830	0.09260271	0.30430694	-0.00973115	0.09269741
3888	3.67943757	0.08869513	0.29781728	-0.00944189	0.08878428
4104	3.67984007	0.08541057	0.29225086	-0.00903938	0.08549228
4320	3.67978144	0.08239413	0.28704378	-0.00909801	0.08247691
4536	3.68011039	0.07948762	0.28193548	-0.00876906	0.07956451
4752	3.68060377	0.07703757	0.27755642	-0.00827568	0.07710605
4968	3.68097242	0.07471847	0.27334679	-0.00790704	0.07478099
5184	3.68087247	0.07264338	0.26952436	-0.00800698	0.07270749
5400	3.68109579	0.07075371	0.26599570	-0.00778366	0.07081430
	(.0022683)	(.0017238)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 17



0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
ESTIMATED BIAS
Unit Exponential .975 Quantile $S_{.975}$=3.688879
Stochastic Approximation With:
Maximum Transformation V=27
2-Fold Jackknife 1/(m**.5) Term
60,000 Replications
w/o Jackknife $\hat{b}_{.975}(m)$
w/ Jackknife $\hat{\hat{b}}_{.975}(m)$
Estimated σ of $\hat{b}_{.975}(200)$ is .00171 (9 df)
Estimated σ of $\hat{\hat{b}}_{.975}(200)$ is .00227 (9 df)
216.0
1512.0
2808.0
4104.0
5400.0
SAMPLE SIZE m

FIGURE 8

UNIT EXPONENTIAL .999 QUANTILE: $S_{\alpha=.999}=6.907755$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=700
WITHOUT JACKKNIFE 60,000 REPLICATIONS

m	$E(\hat{S}_\alpha)$	$VAR(\hat{S}_\alpha)$	$STD\ DEV(\hat{S}_\alpha)$	$BIAS(\hat{S}_\alpha)$	MSE
5600	6.95313711	0.31538654	0.56159286	0.04538183	0.31744605
11200	6.93550984	0.18611215	0.43140718	0.02775456	0.18688247
16800	6.92973056	0.13945707	0.37343951	0.02197528	0.13993998
22400	6.29406434	0.11622209	0.34091361	0.01630906	0.11648808
28000	6.92204279	0.10185784	0.31915174	0.01428751	0.10206197
33600	6.91982085	0.09148774	0.30246941	0.01206557	0.09163332
39200	6.91852010	0.08427502	0.29030160	0.01076482	0.08439090
44800	6.91773864	0.07836587	0.27993904	0.00998336	0.07846553
50400	6.91652646	0.07397515	0.27198372	0.00877118	0.07405208
56000	6.91585850	0.07024125	0.26503065	0.00810332	0.07030691
61600	6.91507908	0.06701421	0.25887103	0.00732380	0.06706785
67200	6.91441978	0.06440668	0.25378471	0.00666451	0.06445109
72800	6.91373058	0.06209137	0.24918139	0.00597530	0.06212707
78400	6.91289783	0.06000825	0.24496581	0.00514256	0.06003470
84000	6.91302871	0.05822901	0.24130688	0.00527343	0.05825682
89600	6.91261125	0.05653701	0.23777513	0.00485597	0.05656059
95200	6.91239268	0.05510953	0.23475419	0.00463740	0.05513103
100800	6.91226693	0.05378367	0.23191307	0.00451165	0.05380403
106400	6.91203576	0.05258662	0.22931773	0.00428048	0.05260494
112000	6.91171632	0.05157640	0.22710439	0.00396104	0.05159209
117600	6.91157911	0.05060444	0.22495430	0.00382383	0.05061906
123200	6.91165251	0.04969747	0.22292930	0.00389723	0.04971266
128800	6.91157643	0.04884637	0.22101216	0.00382116	0.04886098
134400	6.91122833	0.04807308	0.21925575	0.00347305	0.04808515
140000	6.91115469	0.04738551	0.21768213	0.00339941	0.04739707
	(.0019035)	(.0018588)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 18

UNIT EXPONENTIAL .999 QUANTILE: $S_{\alpha=.999}=6.907755$
STOCHASTIC APPROXIMATION WITH MAXIMUM TRANSFORMATION V=700
WITH JACKKNIFE $m^{-.5}$ TERM 60,000 REPLICATIONS

m	$E(\tilde{\tilde{S}}_\alpha)$	$VAR(\tilde{\tilde{S}}_\alpha)$	STD DEV$(\tilde{\tilde{S}}_\alpha)$	BIAS$(\tilde{\tilde{S}}_\alpha)$	MSE
5600	6.91449402	1.77862943	1.33365266	0.00673874	1.77867484
11200	6.89302696	0.78420640	0.88555429	-0.01472832	0.78442332
16800	6.89385469	0.48561450	0.69686046	-0.01390059	0.48580773
22400	6.89092571	0.35437933	0.59529768	-0.01682957	0.35466256
28000	6.89359768	0.28049224	0.52961518	-0.01415760	0.28069267
33600	6.89382694	0.23232246	0.48199841	-0.01392834	0.23251646
39200	6.89506346	0.20038706	0.44764613	-0.01269182	0.20054814
44800	6.89618498	0.17681741	0.42049662	-0.01157030	0.17695128
50400	6.89669848	0.15895010	0.39868546	-0.01105680	0.15907235
56000	6.89740806	0.14500201	0.38079130	-0.01034722	0.14510908
61600	6.89784269	0.13354647	0.36544011	-0.00991259	0.13364473
67200	6.89815921	0.12448057	0.35281804	-0.00959607	0.12457266
72800	6.89820942	0.11641109	0.34119070	-0.00954586	0.11650222
78400	6.89801246	0.10945554	0.33084066	-0.00974282	0.10955046
84000	6.89917601	0.10340189	0.32156164	-0.00857927	0.10347549
89600	6.89938039	0.09819057	0.31335375	-0.00837489	0.09826071
95200	6.89974253	0.09378814	0.30624849	-0.00801274	0.09385234
100800	6.90020235	0.08983334	0.29972211	-0.00755293	0.08989039
106400	6.90036611	0.08643043	0.29399053	-0.00738917	0.08648503
112000	6.90048983	0.08338944	0.28877230	-0.00726545	0.08344223
117600	6.90103202	0.08055786	0.28382716	-0.00672326	0.08060306
123200	6.90157715	0.07795434	0.27920305	-0.00617813	0.07799251
128800	6.90193323	0.07566786	0.27507792	-0.00582204	0.07570176
134400	6.90177273	0.07349956	0.27110803	-0.00598255	0.07353535
140000	6.90199800	0.07162838	0.26763478	-0.00575728	0.07166152
	(.0031382)	(.0017436)			

Quantities in brackets are estimates of the standard deviations of the estimates.

TABLE 19

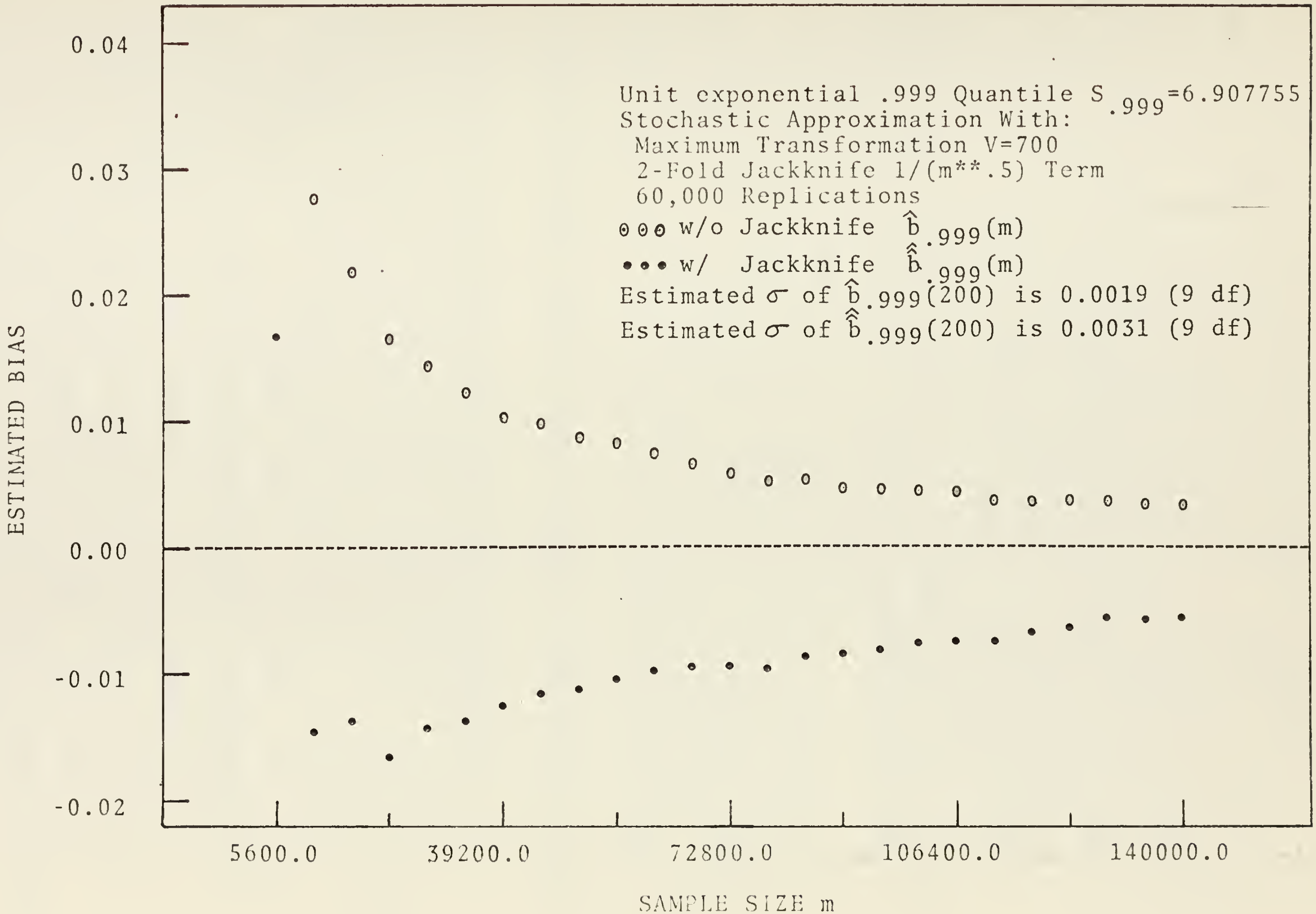

0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
ESTIMATED BIAS
5600.0
39200.0
72800.0
106400.0
140000.0
SAMPLE SIZE m
Unit exponential .999 Quantile $S_{.999}$=6.907755
Stochastic Approximation With:
Maximum Transformation V=700
2-Fold Jackknife 1/(m**.5) Term
60,000 Replications
w/o Jackknife $\hat{b}_{.999}(m)$
w/ Jackknife $\hat{\hat{b}}_{.999}(m)$
Estimated σ of $\hat{b}_{.999}(200)$ is 0.0019 (9 df)
Estimated σ of $\hat{\hat{b}}_{.999}(200)$ is 0.0031 (9 df)

FIGURE 9

In each case the jackknife estimate was negatively biased as predicted from equation 4.14. However, evaluation of the 0.5 quantile data (Table 11) indicates that the resulting rate of convergence to be on the order of $m^{-.289}$. Further, successive regressions on the points in the tail of the curve indicate that the value of γ is decreasing, possibly to $\gamma = 0.25$. Also, the sample bias obtained for the 0.5 quantile goes negative at a sample size value between 192 and 200 which negates the assumption that the coefficients in equation 3.4 are positive.

The results support the conjecture that a $m^{-.5}$ term was significant in the bias. However, it also appears that there exists a lower order term, probably of the order $m^{-.25}$. This would imply that the bias may be of the form

$$\text{bias} = am^{-.25} + bm^{-.5} + cm^{-.75} + dm^{-1} + 0(m^{-1.25}). \quad (7.3)$$

A weighted linear regression using this equation was performed on the data for the unit exponential $\alpha = 0.5$ quantile without jackknife listed in Table 10. The resulting estimates for the leading four terms were $\hat{a} = -0.0501824$, $\hat{b} = 0.0251160$, $\hat{c} = 0.790984$, and $\hat{d} = -0.667208$. The values of the observed bias, bias estimated from the equation fit by the weighted linear regression and the residuals are listed in Table 20. The equation fit achieved a multiple correlation coefficient squared of .9999. Note that the simulation for this term is very large (390,000 replications) so that such a large value of R^2 is not surprising.

Results of the weighted linear regression on the estimated bias of the stochastic approximation with maximum transformation without jackknife for the .500 quantile of the unit exponential distribution.

Equation fit is: $\hat{b}=-.05018m^{-.25}+.02511m^{-.5}+.7909m^{-.75}-.6672m^{-1}$.

m	OBSERVED BIAS	FITTED BIAS	RESIDUALS
8	0.06185200	0.061924000	-0.0000721790
16	0.03858600	0.038360000	0.0002255200
24	0.02771000	0.027601000	0.0001085100
32	0.02091300	0.021281000	-0.0003677000
40	0.01695000	0.017067000	-0.0001171700
48	0.01407700	0.014035000	0.0000421500
56	0.01176400	0.011736000	0.0000272330
64	0.01004700	0.009929100	0.0001174000
72	0.00846910	0.008467200	0.0000019232
80	0.00726480	0.007258400	0.0000064214
88	0.00630860	0.006241000	0.0000675780
96	0.00540630	0.005372200	0.0000340760
104	0.00461510	0.004621200	-0.0000060859
112	0.00388570	0.003965200	-0.0000794790
120	0.00335990	0.003387000	-0.0000271450
128	0.00290560	0.002873600	0.0000320310
136	0.00246430	0.002414400	0.0000498980
144	0.00203060	0.002001300	0.0000292840
152	0.00165660	0.001627700	0.0000289150
160	0.00126970	0.001288100	-0.0000184060
168	0.00095590	0.000978130	-0.0000222310
176	0.00070190	0.000694060	0.0000078395
184	0.00046030	0.000432790	0.0000275070
192	0.00015200	0.000191710	-0.0000397060
200	-0.00008800	-0.000031430	-0.0000565700

TABLE 20

If the conjecture that the bias is of the form in 7.3 is true, the bias computed as a result of the jackknife for the $m^{-.5}$ term would be

$$\text{bias} = +.544am^{-.25} - .648cm^{-.75} - 1.41dm^{-1} + 0(m^{-1.25}), \quad (7.4)$$

which is consistent with the results obtained.

A jackknife for the $m^{-.25}$ term would result in a bias of the form

$$\text{bias} = -1.19bm^{-.5} - 2.609cm^{-.75} - 4.29dm^{-1} + 0(m^{-1.25}), \quad (7.5)$$

which would indicate that the magnitude of the resulting bias may be a large negative value and converge as $m^{-.5}$. In fact, a pilot run to jackknife the $m^{-.25}$ term gave results with a large negative bias and converging approximately as $m^{-.5}$. These results are plotted in Figure 10. The pilot run consisted of 120,000 replications for the $\alpha = .5$ quantile of the unit exponential distribution.

An informal analysis of these results indicated that the rate of convergence was on the approximate order of $m^{-.5}$. These results appear to conform fairly well with the results predictable from equation 7.5. It would therefore appear that the form of the bias of the stochastic approximation with maximum transformation may be of the form in equation 7.3.

If the form of the bias is in fact of the form in 7.3, a possible approach would be to jackknife for both the $m^{-.5}$ and $m^{-.25}$ terms. This could be accomplished by using

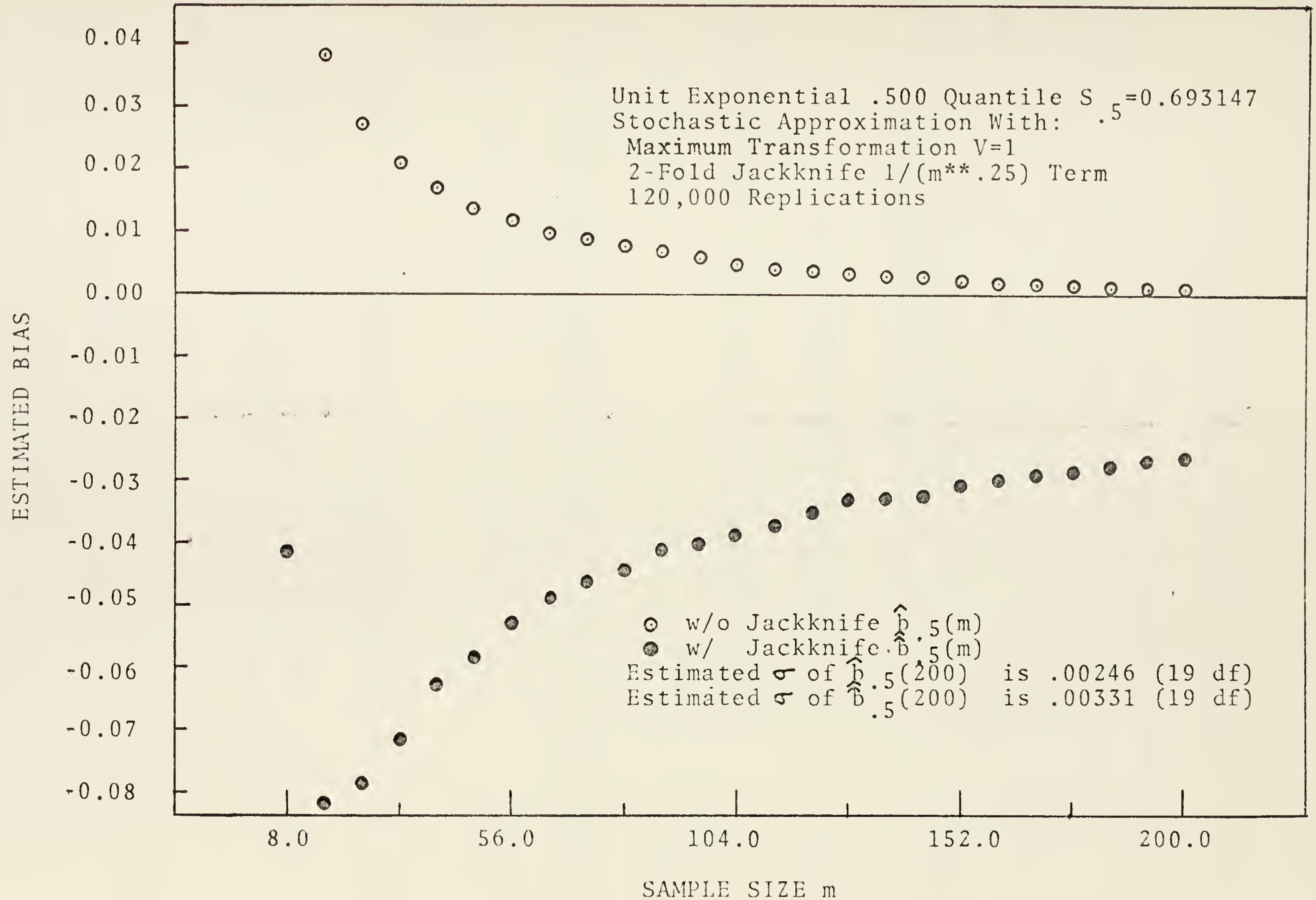

Unit Exponential .500 Quantile S .5=0.693147
Stochastic Approximation With:
Maximum Transformation V=1
2-Fold Jackknife 1/(m**.25) Term
120,000 Replications
w/o Jackknife b̂ .5(m)
w/ Jackknife b̂̂ .5(m)
Estimated σ of b̂ .5(200) is .00246 (19 df)
Estimated σ of b̂̂ .5(200) is .00331 (19 df)
0.04
0.03
0.02
0.01
0.00
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
ESTIMATED BIAS
8.0
56.0
104.0
152.0
200.0
SAMPLE SIZE m

FIGURE 10

as pseudo-values

$$\tilde{S}_{ij} = A \cdot \hat{S}_\alpha(m) + B \cdot \hat{S}_\alpha(m/2)_i + C \cdot \hat{S}_\alpha(m/4)_j \quad i,j=1,2 \qquad (7.6)$$

where $\hat{S}_\alpha(m)$ and $\hat{S}_\alpha(m/2)_i$ are as previously discussed and $\hat{S}_\alpha(m/4)_{ij}$ is computed by partitioning the ith subset of m/2 values into two disjoint subsets of m/4 values each. Then $\hat{S}_\alpha(m/4)_{ij}$ is the estimate based on the values in the jth subset from the ith half sample set.

The jackknifed estimate then becomes

$$\tilde{\tilde{S}}_\alpha = \frac{[\tilde{S}_{11} + \tilde{S}_{12} + \tilde{S}_{21} + \tilde{S}_{22}]}{4} \qquad (7.7)$$

with the result that if A, B, and C are chosen as

$$A = 1-B-C, \qquad (7.8)$$

$$B = -(1 + C\sqrt{2}-1)/(2^{-.25} - 1), \qquad (7.9)$$

$$C = (\sqrt{2}-2^{-.25})/[2^{-.25} - 1 - (\sqrt{2}-1)^2], \qquad (7.10)$$

then

$$E(\tilde{\tilde{S}}_\alpha) - S_\alpha = -1.68cm^{-.75} + 6.06dm^{-1} + 0(m^{-1.25}), \qquad (7.11)$$

which is void of the $m^{-.25}$ and $m^{-.5}$ bias terms. This may, of course, inflate the variance of the estimator to an unacceptable level.

In conclusion, an assessment of the utility of the method for obtaining an extreme quantile discussed in this thesis can be made by referring to Figures 11 and 12. These show the standard deviations and root mean square errors of the

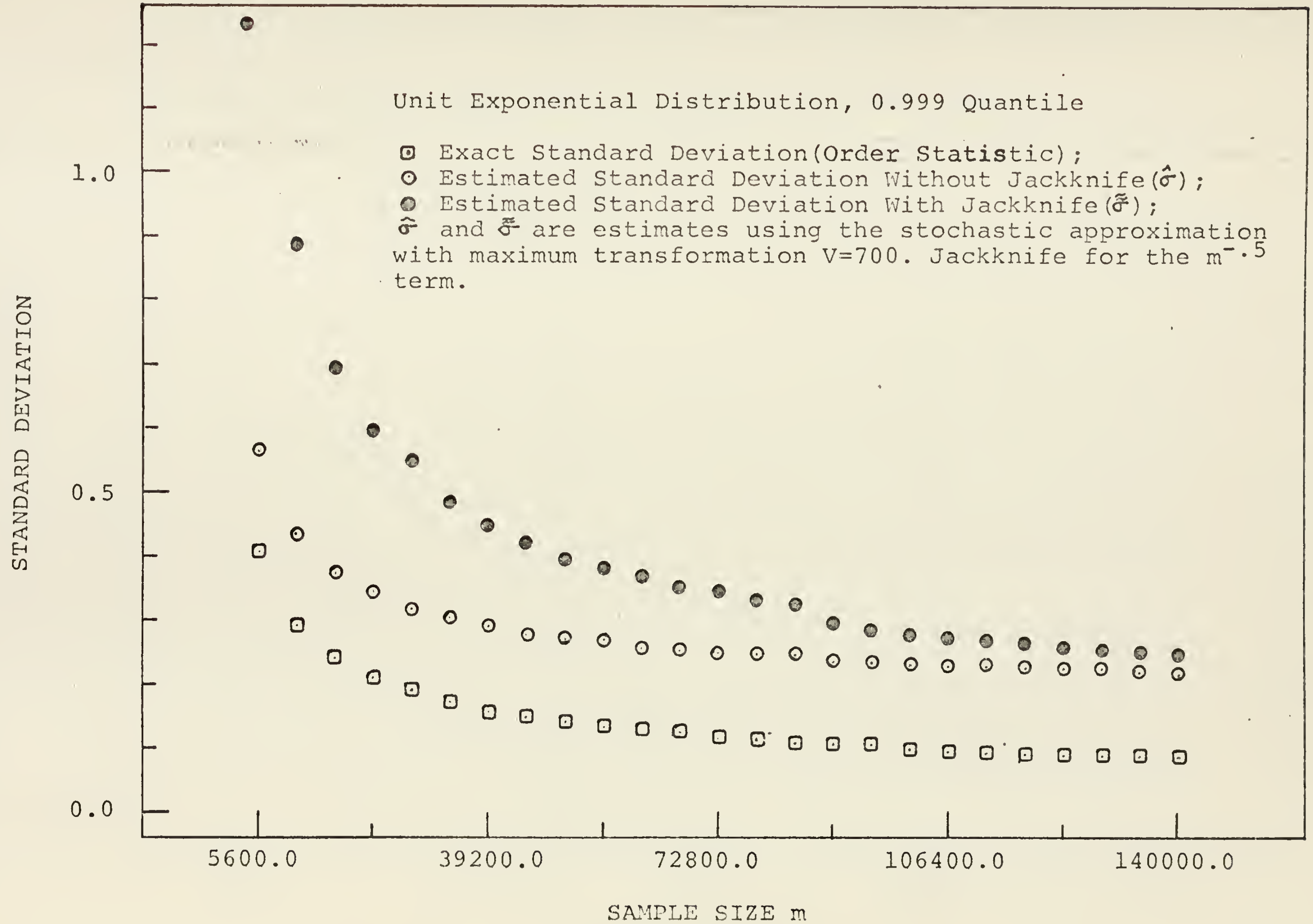

Unit Exponential Distribution, 0.999 Quantile
Exact Standard Deviation(Order Statistic);
Estimated Standard Deviation Without Jackknife($\hat{\sigma}$);
Estimated Standard Deviation With Jackknife($\tilde{\sigma}$);
$\hat{\sigma}$ and $\tilde{\sigma}$ are estimates using the stochastic approximation with maximum transformation V=700. Jackknife for the $m^{-.5}$ term.
STANDARD DEVIATION
1.0
0.5
0.0
5600.0
39200.0
72800.0
106400.0
140000.0
SAMPLE SIZE m

FIGURE 11

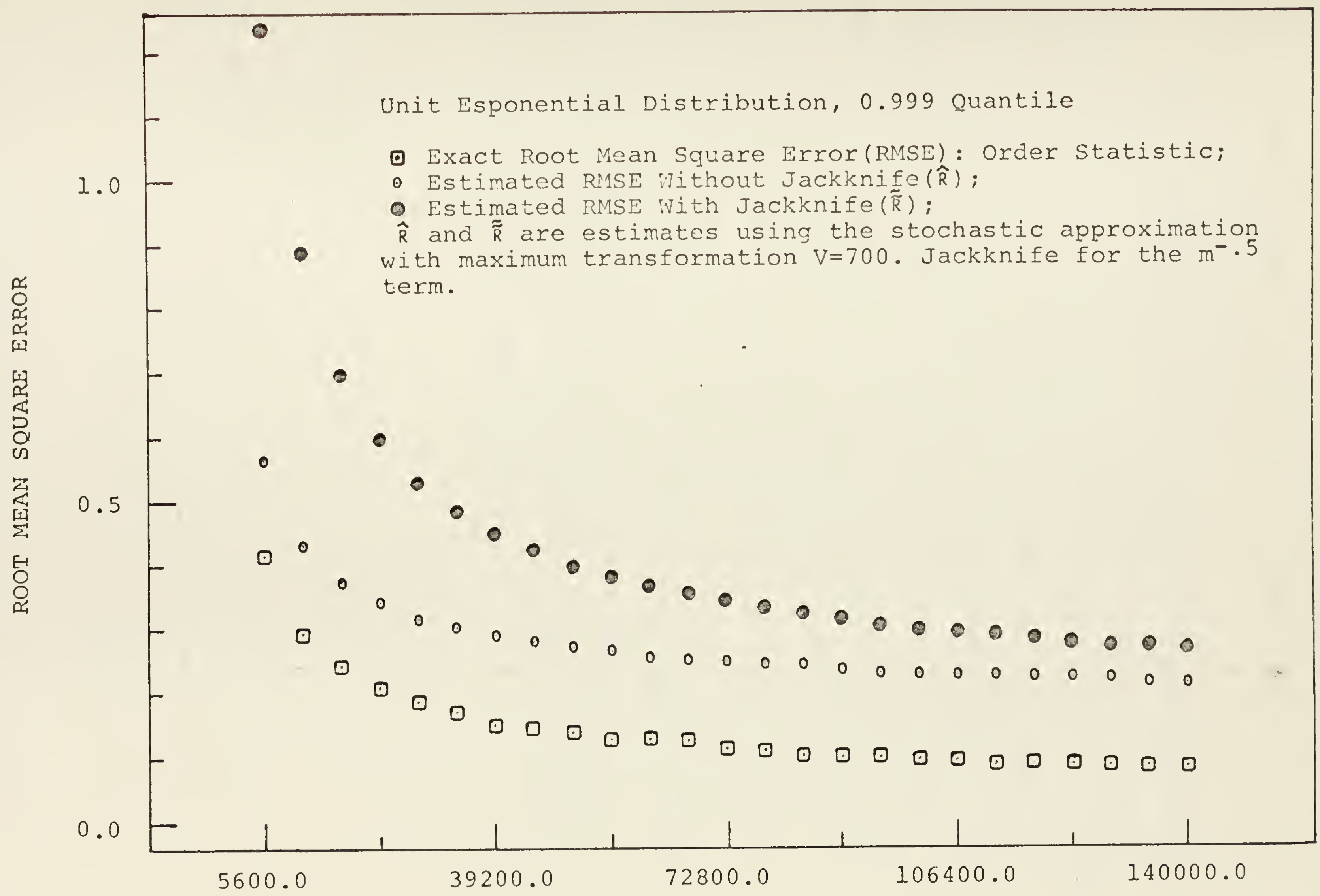

Unit Esponential Distribution, 0.999 Quantile
Exact Root Mean Square Error(RMSE): Order Statistic;
Estimated RMSE Without Jackknife($\hat{R}$);
Estimated RMSE With Jackknife($\tilde{\tilde{R}}$);
$\hat{R}$ and $\tilde{\tilde{R}}$ are estimates using the stochastic approximation with maximum transformation V=700. Jackknife for the $m^{-.5}$ term.
ROOT MEAN SQUARE ERROR
1.0
0.5
0.0
5600.0
39200.0
72800.0
106400.0
140000.0
SAMPLE SIZE m

FIGURE 12

order statistic estimate, stochastic approximation estimate with maximum transformation (with and without jackknife) as a function of the sample size.

There is clearly an increase in the standard deviation of the quantile estimate, for all m, as one goes from the estimate based on order statistics to the stochastic approximation estimate and then to the stochastic approximation with jackknife. However, this is more than offset by the (fixed) smaller memory requirements for the stochastic approximation estimate and the fact that the time needed to obtain an estimate from a sample of size m is proportional to m, while the order statistic estimate requires memory proportional to m and time (to order the data) proportional to mlog(m). These factors become even more critical when considered in the context of estimating large numbers of quantiles or in cases where m is large (see Goodman, Lewis, Robbins, 1973).

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ABSTRACT

The rate of convergence of the expected value of quantile estimates using a stochastic approximation with the maximum transformation is evaluated. The analysis is performed using linear regression techniques on computer simulation results for quantile estimates of the unit exponential distribution. Included is a discussion of the use of the jackknife technique to reduce the bias of the stochastic approximation quantile estimates. Simulation results for the 2-fold jackknife for the $m^{-.5}$ term are tabulated. The main conclusion of the analysis is that the lowest order term in the expression for the expected value of the estimate as a function of sample size decreases as $m^{-.25}$.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Simulation						
Estimation						
Quantile						
Stochastic Approximation						
Order Statistics						

Thesis 143332
Y845 Yuguchi
c.1 Quantile estimation using the maximum transformation, stochastic approximation and the jackknife.